Basics of M-Theory

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Abstract

This is a review article of eleven dimensional supergravity in which we present all necessary calculations, namely the Noether procedure, the equations of motion (without neglecting the fermions), the Killing spinor equation, as well as some simple and less simple supersymmetric solutions to this theory. All calculations are printed in much detail and with explicit comments as to how they were done. Also contained is a simple approach to Clifford algebras to prepare the grounds for the harder calculations in spin space and Fierz identities.

Contents

1 Preface

This is a provisional version of lecture notes on M-theory. The etymology of the name "M-Theory" is explained in [1] and traced back to (M)embranes. Nowadays the "M" is thought to refer to the word M(other) due to the pivotal role it seems to claim in the unification of string theories. Strictly speaking, M-Theory is as yet undiscovered. All that is known is its low energy limit, i.e. eleven dimensional supergravity (11d SUGRA).

The construction of eleven dimensional supergravity was performed in 1978 [2]. In this seminal paper the Lagrangian density, the equations of motion and the transformation properties of the fields with respect to the supercharges were obtained by performing the Noether procedure. Later the doubled field approach was used to rederive these results from a suitably chosen supergroup by cohomologial techniques [3]. A new investigation of the formal structure of eleven dimensional supergravity was pursued in two papers twenty years later [4, 5]. By combining two different techniques, i.e. the doubled field approach and the method of nonlinear realisations, it became possible to understand the equations of motion of the bosonic p-form fields as generalised self duality conditions. The field strengths can be obtained from algebraic considerations. This idea can be extended to also include gravity thereby identifying a huge symmetry algebra as was first laid out in [6, 7]. Apparently, this approach can be extended to more supergravity theories [8] and possibly be used for an a posteriori understanding of some simple solutions [9]. Supposedly, these new symmetries can be used to further our insight into M-theory. However, those recent developments shall not be part of this lecture.

Much is known about eleven dimensional supergravity, but it seems hard to find most of the calculations in a *single* article using the *same* conventions throughout. Hence, it is our intension to give a readable account of all typical calculations concerning eleven dimensional supergravity in a single work. A detailed discussion of the equations of motion (section 3), the derivation of the Lagrangian, the supervariations (section 4) and the Killing spinor equation (section 5) is presented. A similarly detailed discussion of the most typical supersymmetric solutions is given as well (section 5, 6, and 7). Hereby, we put weight on going through the calculations in full length. Some easy parts of these calculations are given as homeworks. The reader interested in more general questions on p-brane solutions is referred to [10, 11].

The claimed pivotal role of M-theory is due to the established relations to the ten dimensional supergravity theories that are low energy limits of superstring theory. The advantage of the eleven dimensional theory over its ten dimensional siblings is its uniqueness and its simplicity - it contains only three different particles: the graviton, the gravitino, and a gauge potential. A particularly simple relation to ten dimensional IIA supergravity is given by dimensional reduction on a circle. This reduction is also explicitly treated in this paper (section 8). In general, Kaluza Klein reductions are an involved subject since one has to decide whether the lower dimensional theory is consistent. This is always the case for circle compactifications. The more general case is discussed in [12] or the very understandable article by Chris Pope [13].

As for the prerequisites, we merely assume the readers familiarity with the tensor calculus of general relativity. Good and legible introductions into general relativity are [14, 15]. In addition, we assume some general knowledge of the basic concepts underlying supersymmetry. A valuable introduction to supersymmetry is [16]. Good sources of information about supergravity in four dimensions

are [17, 18]. There are more reviews on higher dimensional supergravities. A very useful one is [19].

When discussing supersymmetric solutions to supergravity theories one will immediately come across the topics of calibrations [20, 21, 22] and G-structures [23, 24]. Here these topics are omitted entirely but they will be treated in a later article [25] by one of us.

Since we do not wish to omit fermions (as is typical in SUGRA calculations) more than necessary, we start with a simple introduction into Clifford algebras (section 2) - the essential tool for dealing with fermions and their transformation properties[*1]. This makes our review essentially self contained. Again our presentation of Clifford algebras is both fairly explicit and intuitive.

We would appreciate any comments, suggestions, etc.

[*1] An excellent review of Clifford algebras in general is [26].

2 Clifford Algebras

We will derive Clifford algebras by looking at representations of typical rotation groups arising in mathematical physics.

2.1 An explicit example: $\mathrm{Cliff}(\mathbb{R}^3)$

It is clear that the usual scalar product of $\mathbb{R}^3$ with metric tensor $\eta_{ab} = \mathrm{diag}(1,1,1)$ possesses the invariance group $SO(3)$. This is the group of rotation and hence preserves the length of a vector. From a formal point of view one can define $SO(3)$ as the group of transformations preserving the symmetric bilinear form $B(\circ,\circ)$, defined by

$$B(\vec{x},\vec{y}) \quad = \quad \eta_{ab}\, x^a\, y^b\,. \tag{2.1}$$

Assuming A to be a rotation matrix, the preservation of the bilinear form can be written as

$$(A\vec{x},\, A\vec{y}) = (\vec{x},\, \vec{y})\,. \tag{2.2}$$

The left hand side can also be rewritten as

$$(A\vec{x},\, A\vec{y}) = (\vec{x},\, A^T\, A\,\vec{y})\,, \tag{2.3}$$

and it therefore follows simply that

$$A^T\, A \equiv 1 \quad \rightarrow \quad A^T = A^{-1}\,, \tag{2.4}$$

which is the condition that A has to be an orthogonal matrix. Using $\det(A^T) = \det(A)$, the condition on the determinant of A can be found to be $\det(A)^2 = 1$ which leaves $\det(A) = \pm 1$. Choosing the positive sign entails preservation of orientation which is typically the case for rotations. Allowing both signs, the more general group is called $O(3)$ rather than $SO(3)$.

An obvious question is whether there are more general bilinear forms possessing $SO(3)$ as part of their invariance group. To find such a generalised form we introduce a formal product $\circ$ and define a new symmetric bilinear form just by

$$V(\vec{x},\vec{y}) \quad = \quad \sum_{i\leq j} x^i\, y^j\, \left(e_i \circ e_j + e_j \circ e_i\right). \tag{2.5}$$

This obviously contains B

$$V(\vec{x},\vec{y}) \quad = \quad 2\underbrace{\sum_i x^i\, y^i\, e_i \circ e_i}_{(1)} + \underbrace{\sum_{i<j} x^i\, y^j\, \left(e_i \circ e_j + e_j \circ e_i\right)}_{(2)}, \tag{2.6}$$

the first part can be identified with eq. (2.1) by letting $e_i \circ e_i = B(e_i, e_i) \cdot \mathbb{1}$. The unit matrix has been added to give us more freedom: in particular using the $\circ$ product of two basis vectors does not result in a scalar. With this product the basis vectors form a closing algebra (so any product is well-defined). Any algebra by definition contains a unit element. In the above product we have hence defined that the $\circ$ product of a basis vector with itself be proportional to unity and the factor of proportionality is a scalar, namely the scalar product or the length of the relevant basis vector.
In order to ensure that $SO(3)$ is still part of the invariance group of $V(\vec{x}, \vec{y})$ the second part must vanish. This is achieved by

$$e_i \circ e_j + e_j \circ e_i \quad = \quad 0\,. \tag{2.7}$$

Both conditions can be summarised in the defining formula of a Clifford algebra

$$e_i \circ e_j + e_j \circ e_i \quad = \quad 2 \cdot B(e_i, e_j) \cdot \mathbb{1}\,. \tag{2.8}$$

Let us compare the different products. The usual scalar product of orthogonal basis vectors is of course

$$e_i \cdot e_j = \eta_{ij}\,, \tag{2.9}$$

where η contains -1 or +1 on its diagonal depending on the signature of the underlying $\mathbb{R}^{p,q}$; all other entries are zero.
In relation (2.8) the basis vectors are considered as elements of an algebra - the Clifford algebra. The symmetric combination of the $\circ$ product of two basis vectors vanishes if they are different. Taking the square of a basis element results in an algebra element proportional to unit element.

It is easy to see that the basis elements of the Clifford algebra over $\mathbb{R}^3$ can be taken to be the Pauli matrices, viz.

$$e_1 \mapsto \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad e_2 \mapsto \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad e_3 \mapsto \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{2.10}$$

The full Clifford algebra is generated by all linear combinations and powers of those three elements. The square of all Pauli matrices is the unit matrix which therefore naturally occurs in the Clifford algebra. Hence there are four generators for this Clifford algebra: $\dim(\mathrm{Cliff}(\mathbb{R}^3)) = 4$. In particular, the product of two basis elements in the Clifford algebra is proportional to the remaining basis vector, e.g.

$$e_1 \circ e_2 = i e_3 \quad \text{(and not 0 as in the scalar product)}\,. \tag{2.11}$$

Is the invariance group of $V(\circ,\circ)$ larger than the one of $B(\circ,\circ)$?

As a starting point for answering this questions serves the assumption

$$V(x,\, y) = B(\vec{x},\, \vec{y}) \cdot \mathbb{1}\,. \tag{2.12}$$

Some comments are in place here. On the left hand side of the last equation, there are no arrows above the vectors x, y since x and y are considered as elements of the Clifford algebra. Usually (as on the right hand side) the vector is expanded into the basis vectors by

$$\vec{x} = x^1\, e_1 + x^2\, e_2 + x^3\, e_3\,, \tag{2.13}$$

where basis vectors are multiplied by a scalar product (2.9). On the left hand side of the fore last equation, however, the expansion is

$$x = x^1\, \tau_1 + x^2\, \tau_2 + x^3\, \tau_3\,. \tag{2.14}$$

Hence the "vector" x is expanded into the Clifford algebra and therefore is a two-by-two matrix now. In the bilinear form V the product of two elements using $\circ$ is obtained by usual matrix multiplication. Since V is symmetric, one has to take the symmetrised matrix product as specified in (2.6) to reproduce (2.12).

Since both bilinear forms are supposed to be invariant under rotations, equation (2.12) must also hold in the form

$$V(A\,x,\, A\,y) = B(A\,\vec{x},\, A\,\vec{y}) \cdot \mathbb{1} = B(\vec{x},\, \vec{y}) \cdot \mathbb{1} = V(x,\, y)\,. \tag{2.15}$$

Again, it is easy to understand how the rotation matrix $A \in SO(3)$ acts on the vectors $\vec{x}$ or $\vec{y}$ in this equation: simply in the fundamental representation of $SO(3)$ in terms of three-by-three matrices. The leftmost side is more subtle, since here an action on two-by-two matrices x and y is required which necessarily has to leave the invariance property intact. Since it is known that the **adjoint** representation of $SU(2)$ is three dimensional it occurs natural to try $g\,\tau_i\,g^{-1} = A\,e_i$, where $g \in SU(2)$ is a two-by-two matrix and can therefore easily 'rotate' the Pauli matrices as does the $SO(3)$ matrix A with the basis vectors e_i of $\mathbb{R}^3$.
More explicitly: it is possible to expand any vector into linear combinations of basis vectors of $\mathbb{R}^3$ (2.13). Under $SO(3)$ rotations these basis vectors transform in the obvious way leaving the bilinear form invariant $B(e_i,\, e_j) = B(A\,e_i,\, A\,e_j)$. Using the other symmetric bilinear form in which the vector takes the form of a two-by-two matrix the same invariance can be obtained by using the fact that the adjoint representation of $SU(2)$ is three dimensional and acts on two-by-two matrices. If the $SO(3)$ rotation matrix A is given, the corresponding element $g \in SU(2)$ can be calculated via

$$g\,\tau_i\,g^{-1} = A\,e_i\,, \tag{2.16}$$

meaning that the action on the respective basis elements coincide. The last equation looks simple,

but has to be understood algebraically: on the left hand side there is a product of three different matrices while on the right hand side there is a rotation matrix acting on a vector. The point is that the τ_i despite of having a representation as two-by-two matrices from an algebraic point of view are basis vectors in the same way as the e_i. In this way, the above correspondence is precise.

By presenting the whole calculation, we shall see that the invariance group of the symmetric bilinear form $V(x,\, y)$ is twice as big as the one of $B(\vec{x},\, \vec{y})$. We start by computing the left hand side of the last equation. A general group element of $SU(2)$ can be expressed by $g = \exp(a^i\, t_i)$, where the a^i are the relevant coefficients and $t_i = \frac{i}{2}\tau_i$. The factor of $\frac{i}{2}$ is conventional and will be commented on in due course. Obviously it does not in principal spoil the algebraic relation fulfilled by the Pauli matrices τ_i. An obvious change is that in comparison to equation (2.11) the imaginary unit disappears: $\mathfrak{t}_1\,\mathfrak{t}_2 = -\mathfrak{t}_3$. The generators t_i of the Lie algebra $\mathfrak{su}(2)$ are then

$$\mathfrak{t}_1 \;=\; \begin{pmatrix} 0 & \frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} \qquad \mathfrak{t}_2 \;=\; \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \qquad \mathfrak{t}_3 \;=\; \begin{pmatrix} \frac{i}{2} & 0 \\ 0 & -\frac{i}{2} \end{pmatrix}.$$

The three one parameter subgroups generated by these $\mathfrak{t}_1 \ldots \mathfrak{t}_3$ are

$$g_1 \;=\; \begin{pmatrix} \cos(\frac{\theta_1}{2}) & i\,\sin(\frac{\theta_1}{2}) \\ i\,\sin(\frac{\theta_1}{2}) & \cos(\frac{\theta_1}{2}) \end{pmatrix} \tag{2.17}$$

$$g_2 \;=\; \begin{pmatrix} \cos(\frac{\theta_2}{2}) & \sin(\frac{\theta_2}{2}) \\ -\sin(\frac{\theta_2}{2}) & \cos(\frac{\theta_2}{2}) \end{pmatrix} \tag{2.18}$$

$$g_3 \;=\; \begin{pmatrix} \exp(i\frac{\theta_3}{2}) & 0 \\ 0 & \exp(-i\frac{\theta_3}{2}) \end{pmatrix}. \tag{2.19}$$

It is a straightforward task to calculate the adjoint action of these elements on the basis elements $\mathfrak{t}_i$

$$\begin{aligned} \pi(g_1)\,\mathfrak{t}_1 \;&=\; && \;=\; \mathfrak{t}_1 \\ \pi(g_1)\,\mathfrak{t}_2 \;&=\; \begin{pmatrix} -\frac{i}{2}\sin\theta_1 & \frac{1}{2}\cos\theta_1 \\ -\frac{1}{2}\cos\theta_1 & \frac{i}{2}\sin\theta_1 \end{pmatrix} && \;=\; \cos\theta_1\cdot\mathfrak{t}_2 \;-\; \sin\theta_1\cdot\mathfrak{t}_3 \\ \pi(g_1)\,\mathfrak{t}_3 \;&=\; \begin{pmatrix} \frac{i}{2}\cos\theta_1 & \frac{1}{2}\sin\theta_1 \\ -\frac{1}{2}\sin\theta_1 & -\frac{i}{2}\cos\theta_1 \end{pmatrix} && \;=\; \sin\theta_1\cdot\mathfrak{t}_2 \;+\; \cos\theta_1\cdot\mathfrak{t}_3 \end{aligned} \tag{2.20}$$

Similarly for g_2 and g_3

$$
\begin{aligned}
\pi(g_2)\,\mathfrak{t}_1 &= \begin{pmatrix} \frac{i}{2}\sin\theta_2 & \frac{i}{2}\cos\theta_2 \\ \frac{i}{2}\cos\theta_2 & -\frac{i}{2}\sin\theta_2 \end{pmatrix} &= \quad \cos\theta_2\cdot\mathfrak{t}_1 \ + \ \sin\theta_2\cdot\mathfrak{t}_3 \\
\pi(g_2)\,\mathfrak{t}_2 &= &= \quad \mathfrak{t}_2 \\
\pi(g_2)\,\mathfrak{t}_3 &= \begin{pmatrix} \frac{i}{2}\cos\theta_2 & -\frac{i}{2}\sin\theta_2 \\ -\frac{i}{2}\sin\theta_2 & -\frac{i}{2}\cos\theta_2 \end{pmatrix} &= -\ \sin\theta_2\cdot\mathfrak{t}_1 \ + \ \cos\theta_2\cdot\mathfrak{t}_3
\end{aligned}
\tag{2.21}
$$

$$
\begin{aligned}
\pi(g_3)\,\mathfrak{t}_1 &= \begin{pmatrix} 0 & \frac{i}{2}\,e^{-i\theta_3} \\ \frac{i}{2}\,e^{i\theta_3} & 0 \end{pmatrix} &= \quad \cos\theta_3\cdot\mathfrak{t}_1 \ + \ \sin\theta_3\cdot\mathfrak{t}_2 \\
\pi(g_3)\,\mathfrak{t}_2 &= \begin{pmatrix} 0 & \frac{1}{2}\,e^{-i\theta_3} \\ -\frac{1}{2}\,e^{i\theta_3} & 0 \end{pmatrix} &= -\ \sin\theta_3\cdot\mathfrak{t}_1 \ + \ \cos\theta_3\cdot\mathfrak{t}_2 \\
\pi(g_3)\,\mathfrak{t}_3 &= &= \quad \mathfrak{t}_3\,.
\end{aligned}
\tag{2.22}
$$

The first set of equation (2.20) can be rewritten in matrix form in the following way

$$
\pi(g_1)\,\vec{\mathfrak{t}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix} \cdot \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix} \cdot \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}\,, \tag{2.23}
$$

which is the explicit form of formula (2.16) and hence provides the relation between the $SU(2)$ and $SO(3)$ representation in their action on the respective basis elements. The relevant formulae for the other basic rotations are

$$
\pi(g_2)\,\vec{\mathfrak{t}} = \begin{pmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} \cdot \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \tag{2.24}
$$

$$
\pi(g_3)\,\vec{\mathfrak{t}} = \begin{pmatrix} \cos\theta_2 & \sin\theta_2 & 0 \\ -\sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \tag{2.25}
$$

These equations show that the adjoint action of $SU(2)$ on the basis of the Clifford algebra is precisely the same as the fundamental action of $SO(3)$ on the vector space basis. By construction it is clear that the $SU(2)$ action leaves invariant the bilinear form $V(x,\,y)$.
However, the invariant groups are of different size since we have to assume that there exist objects which transform in the fundamental of $SU(2)$ (while we have used the adjoint to match with the rotation of vectors). The three one parameter subgroups of $SU(2)$ in (2.17) clearly contain half the angle of the corresponding $SO(3)$ rotation though. An object that rotates around 360 degrees in the

fundamental representation of $SU(2)$ will appear like an object that was rotated by 720 degrees in the vector space basis. Or the other way: if we rotate a vector around 360 degrees in the usual vector space basis, it will have come back to itself. An element transforming in the fundamental of $SU(2)$ in the Clifford algebra (these elements are called spinors) will only appear to have been rotated by 180 degrees. Let's consider two spinors rotated by 90 degrees and 270 respectively. In the corresponding $SO(3)$ rotation they correspond to vectors rotated by 180 and 540 degrees, respectively. However, a vector rotated around 180 degrees is naturally indistinguishable from one that was rotated around 540 degrees (difference 360 degrees) but the spinors are obviously differently oriented. This fact can be referred to by saying that $SU(2)$ is the double cover of $SO(3)$. The point for this argument is that a sign change in the $SU(2)$ element is not transported to the relevant $SO(3)$ element since in the adjoint representation required to calculate the relevant $SO(3)$ element the two signs cancel each other. Two different $SU(2)$ elements thus correspond to the same $SO(3)$ element.

2.2 General dimensions and general signature

The explicit example was given over the real vector space $\mathbb{R}^3$, but the idea of Clifford algebras extends to general vector spaces $\mathbb{R}^{p,q}$ or $\mathbb{C}^n$.
Instead of using τ_i which refer to the Pauli matrices, for the general case the Greek Γ is used to refer to the representation of the Clifford algebra

$$\{\,\Gamma_i,\,\Gamma_j\,\} \;=\; \Gamma_i\,\Gamma_j + \Gamma_j\,\Gamma_i \;=\; 2\,\eta_{ij}\cdot \mathbb{1}. \tag{2.26}$$

However, the Pauli matrices play a crucial role in constructing the Γ-matrices which form representations of Clifford algebras of higher dimensional vector spaces. Since the application will be in physics, we will not discuss the case of Clifford algebras over complex numbers (leave alone quaternions), but concentrate on Clifford algebras over $\mathbb{R}^{p,q}$ still allowing arbitrary signature. The obvious question is how to find concrete matrices Γ_i fulfilling the defining relations of Clifford algebras (2.26).
In order to answer this it is helpful to look at the Clifford algebra over $\mathbb{R}^2$ with $\eta = \mathrm{diag}(1,1)$. The point is that the first two Pauli matrices $\tau_{1,2}$ obviously furnish a representation since they do for the three dimensional case $\mathrm{Cliff}(\mathbb{R}^3)$. The only difference is that the third Pauli matrix does not represent a basis element of the vector space anymore (we have only got two in $\mathbb{R}^2$). It turns out

$$\mathrm{Cliff}(\mathbb{R}^2) = \mathrm{Cliff}(\mathbb{R}^3)\,. \tag{2.27}$$

While in the three dimensional case the third Pauli matrix represents the third basis vector in the Clifford algebra (and hence squares to one because we assumed the metric to be positive definite), in the two dimensional case the third Pauli matrix can be used as a projector (since it squares to one). The important role of this projector will be discussed later when we introduce the notion of chirality of spinors.
This last observation is generalisable in the sense that always the Clifford algebra over an even-dimensional real field is the same as the one over the real field in one dimension higher

$$\mathrm{Cliff}(\mathbb{R}^{2k}) = \mathrm{Cliff}(\mathbb{R}^{2k+1})\,. \tag{2.28}$$

Since this has worked so nicely, it might be tempting to try to get the Clifford algebra over $\mathbb{R}^4$ by multiplying all basis elements over $\mathbb{R}^3$, i.e. introduce $e_4 \sim e_1 \circ e_2 \circ e_3$ as an independent basis vector. As demonstrated above, the $\circ$ product turns into matrix multiplication if the basis elements are represented by Pauli matrices. The product of all Pauli matrices is proportional to the unit matrix and hence does not result in a new independent element of the algebra. This is also obvious from the fact that the two-by-two matrices have four degrees of freedom only and can therefore be expanded as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sum_{i=0}^{3} a_i \tau_i \,, \tag{2.29}$$

where we have defined $\tau_0 = \mathbb{1}$. This is a complete basis of the two-by-two matrices, so there is not space for an independent basis element. Upshot: the Clifford algebra over $\mathbb{R}^4$ cannot be represented in two-by-two matrices.

Look at the following expressions involving tensor products of Pauli matrices (to increase the degrees of freedom)

$$\Gamma_1 \quad = \quad Id \otimes \tau_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \Gamma_2 = Id \otimes \tau_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \tag{2.30}$$

$$\Gamma_3 \quad = \quad \tau_1 \otimes T = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \Gamma_4 = \tau_2 \otimes T = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \tag{2.31}$$

where $T = i\tau_1\tau_2$. It is easy to check that

$$\Gamma_1^2 = Id \otimes \tau_1 \cdot Id \otimes \tau_1 = Id^2 \otimes \tau_1^2 = Id \otimes Id = Id_{4\times 4} \,, \tag{2.32}$$

and similarly all other Γ's square to the four dimensional unit matrix (you can check by using the matrix representation and multiply four-by-four matrices explicitly).
Let us check anticommutators

$$\{\Gamma_1, \Gamma_2\} = 2\, Id \otimes (\tau_1\tau_2 + \tau_2\tau_1) = 0 \qquad \text{relying on the two dim Clifford algebra}\,. \tag{2.33}$$

Also very simple are

$$\{\Gamma_1, \Gamma_3\} \quad = \quad 2\tau_1 \otimes (\tau_1 T + T\tau_1) = 2\tau_1 \otimes (\tau_1 T - \tau_1 T) = 0 \quad \text{merely using 2d properties} \tag{2.34}$$

$$\{\Gamma_1, \Gamma_4\} \quad = \quad 2\tau_2 \otimes (\tau_1 T + T\tau_1) = 0 \qquad \text{merely using 2d properties}\,. \tag{2.35}$$

Similarly -try as an exercise- all other anticommutators of different Γ_i, where $i = 1, \ldots, 4$, vanish. In fact, the matrices given above form a representation of Cliff$(\mathbb{R}^4)$, and all that was required are the anticommutation relations of Pauli matrices.
A result previously derived can be checked again. Defining Γ_5 to be proportional to the product of all four dimensional Γ_i, we find

$$\Gamma_5 \sim \Gamma_1 \cdot \Gamma_2 \cdot \Gamma_3 \cdot \Gamma_4 = \tau_1\tau_2 \otimes \tau_1\tau_2 TT = \tau_1\tau_2 \otimes \tau_1\tau_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad (2.36)$$

which squares to unity without being proportional to the unit element itself. Γ_5 is linearly independent and hence is used to construct Cliff$(\mathbb{R}^5)$ which coincides with Cliff$(\mathbb{R}^4)$.
Please note that we have discussed the case of positive definite metric. If indefinite metrics are considered some of the basis elements have to be dressed with factors of i when represented by matrices in the Clifford algebra. For example, if we want to consider the Clifford algebra Cliff$(\mathbb{R}^{1,3})$ with signature $(-,+,+,+)$ then all we need to do is to use $\tilde{\Gamma}_1 = i\Gamma_1$. This does not change the anticommutators but it does change the square $\tilde{\Gamma}_1^2 = -1$. In order to preserve $\Gamma_5^2 = 1$ we need to introduce another imaginary unit $\Gamma_5 = i\tilde{\Gamma}_1 \cdot \Gamma_2 \cdot \Gamma_3 \cdot \Gamma_4$, hence

$$\Gamma_5^2 = -\tilde{\Gamma}_1 \cdot \Gamma_2 \cdot \Gamma_3 \cdot \Gamma_4 \cdot \tilde{\Gamma}_1 \cdot \Gamma_2 \cdot \Gamma_3 \cdot \Gamma_4 = \tilde{\Gamma}_1^2 \Gamma_2 \cdot \Gamma_3 \cdot \Gamma_4 \cdot \Gamma_2 \cdot \Gamma_3 \cdot \Gamma_4 = -\Gamma_2^2 \cdot \Gamma_3 \cdot \Gamma_4 \Gamma_3 \cdot \Gamma_4 = \Gamma_3^2 \cdot \Gamma_4^2,$$

which indeed is the unit matrix.

The idea of taking tensor products of Pauli matrices is generalisable. In fact, it is again easy to check that for a general Clifford algebra over $\mathbb{R}^{2k}$ the basis elements can be defined via

$$e_j \to Id \otimes \ldots \otimes Id \otimes \tau_{\alpha(j)} \otimes \underbrace{T \otimes \ldots \otimes T}_{[\frac{j-1}{2}]-\text{times}}, \qquad (2.37)$$

where

$$T = i \cdot \tau_1\tau_2 \qquad \alpha(j) = 1, \text{if } j \text{ is odd, or} \quad \alpha(j) = 2 \text{ otherwise}.$$

In this way, the properties of the Pauli matrices get inherited to any dimensionality. Let us finally give a representation of the eleven basis elements of Cliff$(\mathbb{R}^{11})$ just to ensure that the above formula looks more complicated than it actually is. The first ten basis elements are simply

$$\begin{aligned} e_1 &\sim Id \otimes Id \otimes Id \otimes Id \otimes \tau_1, & e_2 &\sim Id \otimes Id \otimes Id \otimes Id \otimes \tau_2, \\ e_3 &\sim Id \otimes Id \otimes Id \otimes \tau_1 \otimes T, & e_4 &\sim Id \otimes Id \otimes Id \otimes \tau_2 \otimes T, \\ e_5 &\sim Id \otimes Id \otimes \tau_1 \otimes T \otimes T, & e_6 &\sim Id \otimes Id \otimes \tau_2 \otimes T \otimes T, \\ e_7 &\sim Id \otimes \tau_1 \otimes T \otimes T \otimes T, & e_8 &\sim Id \otimes \tau_2 \otimes T \otimes T \otimes T, \\ e_9 &\sim \tau_1 \otimes T \otimes T \otimes T \otimes T, & e_{10} &\sim \tau_2 \otimes T \otimes T \otimes T \otimes T. \end{aligned} \qquad (2.38)$$

The eleventh basis element is taken to be the product of all other basis elements and hence is

$$e_{11} \sim i\tau_1\tau_2 \otimes \tau_1\tau_2 \otimes \tau_1\tau_2 \otimes \tau_1\tau_2 \otimes \tau_1\tau_2 \,. \tag{2.39}$$

It is straight forward to check that all these elements square to unity (in e_{11} we have introduced the imaginary unit to achieve this) while the anticommutator of different basis elements vanishes. Taking tensor products of five two-by-two matrices,we note that the Clifford algebra over $\mathbb{R}^{11}$ is represented in 32-by-32 matrices. In the next section, the Clifford algebra over $\mathbb{R}^{1,10}$ is constructed in a slightly different way.

To slowly find our way towards calculation involving Clifford algebras it is natural to derive the algebra of products of two Γ matrices which is defined by

$$\Gamma_{ab} = \Gamma_{[a}\,\Gamma_{b]} \,.$$

Please check in detail the following calculation

$$\Gamma_{ab}\Gamma_{cd} = \Gamma_{abcd} - \eta_{ac}\Gamma_{bd} + \eta_{ad}\Gamma_{bc} - \eta_{bd}\Gamma_{ac} + \eta_{b}c\Gamma_{ad} \,,$$

where Γ_{abcd} is the antisymmetrised product of four Γ's. The antisymmetrisation ensures that all indices are different. Since the square of Γ matrices is defined by the Clifford relation to be proportional to unity, only the shortest (simplest) representative of a particular Clifford algebra element is used, for example: instead of writing $\Gamma_1 \cdot \Gamma_1 \cdot \Gamma_2$ we use $\pm\Gamma_2$ since $\Gamma_1 \cdot \Gamma_1 = \pm\mathbb{1}$ (depending on the case).
The last equation expressed in words: either all indices of the product of two Γ_{ab} are different (in which case we get a four indexed Γ matrix, i.e. the first term on the right hand side), or two indices coincide and can be taken to be proportional to the metric (and we have written all possible indices that can coincide). Relabelling indices the last equation takes the form

$$\Gamma_{cd}\Gamma_{ab} = \Gamma_{cdab} - \eta_{ca}\Gamma_{db} + \eta_{cb}\Gamma_{da} - \eta_{ca}\Gamma_{ca} + \eta_{da}\Gamma_{cb} \,.$$

Therefore

$$\begin{aligned} \Gamma_{ab}\Gamma_{cd} - \Gamma_{cd}\Gamma_{ab} = [\Gamma_{ab},\, \Gamma_{cd}] &= \Gamma_{abcd} - \eta_{ac}\Gamma_{bd} + \eta_{ad}\Gamma_{bc} - \eta_{bd}\Gamma_{ac} + \eta_{bc}\Gamma_{ad} \\ &- (\Gamma_{cdab} - \eta_{ca}\Gamma_{db} + \eta_{cb}\Gamma_{da} - \eta_{ca}\Gamma_{ca} + \eta_{da}\Gamma_{cb}) \,. \end{aligned}$$

The last expression can be simplified by considering the antisymmetry in the product of Clifford elements and also using the symmetry in the metric $\eta_{ac} = \eta_{ca}$ to give

$$[\Gamma_{ab},\, \Gamma_{cd}] = -2\eta_{ac}\Gamma_{bd} + 2\eta_{ad}\Gamma_{bc} - 2\eta_{bd}\Gamma_{ac} + 2\eta_{bc}\Gamma_{ad} \,.$$

Defining $\Gamma_{ab} = 2J_{ab}$ (Γ matrices transform with twice the angle) this can be re-expressed as

$$[J_{ab},\, J_{cd}] \;\; = \;\; -\,\eta_{ac} J_{bd} \,+\, \eta_{ad} J_{bc} \,-\, \eta_{bd} J_{ac} \,+\, \eta_{bc} J_{ad}\,, \tag{2.40}$$

which turns out to be the algebra of the rotation generators J_{ab}.

Summary 1. *We have made a full circle and have thus shown that everything is consistent. We have seen that the scalar product can be embedded into an algebra where the basis vectors fulfil slightly generalised relations (the product of two basis elements is non-trivial) but maintain the length of a vector (the square of basis vectors is one - or rather* $\mathbb{1}$ *since in the new algebra). By definition is the product of two different basis elements in the Clifford algebra antisymmetric. We have also just seen that this product of two basis elements except for a factor of two fulfils the algebraic relations typical of rotations. Hence* $\frac{1}{2}\Gamma_{ab}$ *can be considered as generators of rotations.*
Objects transforming under Clifford algebra elements in the fundamental representation, i.e. matrix multiplication, are called spinors. We have argued how to get explicit representations of all Clifford elements in all dimensions and all signatures.

2.3 The Groups PIN and SPIN.

We have derived all relations between the algebras of rotations and spin groups but we will now show that everything can even be understood on the level of the corresponding groups. Hence we define: Be $x \in \mathbb{R}^n$ a vector, and C_n the Clifford algebra with positive definite metric $\eta = diag(1, \ldots, 1)$. Accordingly, $x \cdot x = ||x||^2$ can be used to find an inverse element

$$x^{-1} = \frac{x}{||x||^2}\,.$$

The crucial point is that the basis vectors are naturally elements of a vector space but at the same time also elements of the Clifford algebra. In the Clifford algebra it is also possible to consider all products of basis elements with respect to the Clifford multiplication $\circ$, for example, if Γ_i represents the basis element e_i in the Clifford algebra, then also $\Gamma_{[ij]}$ or $\Gamma_{[ijk]}$ are in the Clifford algebra. In particular the Clifford algebra is generated by all the antisymmetric products of gamma matrices. The highest possible number of indices is the number of basis elements, i.e. the dimensionality of the vector space $\mathbb{R}^n$ under consideration. Adding all possible combinations of any number of indices gives the result

$$\sum_{p=0}^{n} \binom{n}{p} = (1+1)^n = 2^n\,. \tag{2.41}$$

In even dimensions these 2^n elements are linearly independent. As was demonstrated above, the odd dimensional case has the same dimensionality as the next lower even dimensional case since the product of all basis elements just forms the representative for the additional direction (for example $\Gamma_3 \sim \Gamma_{12}$). We note that in principle this number of degrees of freedom could be nicely represented

by $2^{\lfloor\frac{n}{2}\rfloor}$-by-$2^{\lfloor\frac{n}{2}\rfloor}$ matrices. These would naturally act on vectors (better spinors) with dimension $2^{\lfloor\frac{n}{2}\rfloor}$. This is indeed true for most of the cases.

We are ready to define

Definition 1. *$PIN(n) \subset C_n$ is the group which is multiplicatively generated by all vectors $x \in S^{n-1}$ (sphere).*

Before moving on to the definition of the $SPIN(n)$ group, we want to motivate its construction by considering the involution on the vector space $\mathbb{R}^n$ which simply changes the sign of all vectors $V \ni v \to -v \in V$. In the Clifford algebra this means that all elements generated by an even number of basis elements are inert under this mapping, but not the vectors themselves. Nor are the elements inert that are generated by an odd number of basis elements. The $PIN(n)$-group is spanned by all multiplicatively generated elements independently of whether the number of basis elements is even or odd. The $SPIN(n)$ group is defined to be the subgroup of $PIN(n)$ which is inert under the sign swap of vectors

Definition 2. *$SPIN(n) \subset C_n = PIN(n) \cap C_0$, where C_0 is inert under sign swaps of single basis elements.*

An anti-involution γ can be defined by inverting the order of the basis elements:

$$\gamma(x_1 \circ \ldots \circ x_m) \;=\; \gamma(x_m) \circ \ldots \circ \gamma(x_1) \quad \& \quad \gamma(x_j) = x_j, \quad \forall\, x_j \in \mathbb{R}^n \;. \tag{2.42}$$

γ can be understood as the inversion in $PIN(n)$, e.g.

$$\gamma(e_i \circ e_j \circ e_k \circ e_l) \;=\; e_l \circ e_k \circ e_j \circ e_i \,, \tag{2.43}$$

since now the square simply reads

$$(e_i \circ e_j \circ e_k \circ e_l) \circ (e_l \circ e_k \circ e_j \circ e_i) \equiv \mathbb{1}\,, \tag{2.44}$$

which is obviously $\mathbb{1}$. γ can be used to define reflexions and is used to define the adjoint representation of $PIN(n)$ (cf. (2.16)):

Lemma 1. *If $y \in \mathbb{R}^n \subset C_n$ and $x \in PIN(n) \subset C_n$, then $x \cdot y \cdot \gamma(x)$ is again in $\mathbb{R}^n \subset C_n$.*

To see this we can restrict ourselves to the case of one basis element of the PIN-group $x = e_1$; all other cases can be derived analogously. A simple calculation with a general vector $y = \sum_1^n y_i e_i$ yields

$$\begin{aligned} x \cdot y \cdot \gamma(x) \;&=\; e_1(\sum_1^n y_i e_i)e_1 = e_1 y_1 e_1 e_1 - \sum_2^n y_i e_i \\ &=\; y_1 e_1 - \sum_2^n y_i e_i, \end{aligned}$$

where we have merely used $e_1^2 = 1$ as defined from the quadratic form. This, however, is obviously a reflexion of the vector y. It is easy to see that taking a composite element of PIN will only result in successive reflexions.
Correspondingly, $SPIN$ will necessarily generate an even number of reflexions. It is worthwhile to state an old fact which will bring out the full significance of Clifford algebras in physics: EVERY ROTATION CAN BE UNDERSTOOD AS AN EVEN NUMBER OF REFLEXIONS. We have come a long way from abstract mathematics down to rotations and reflexions in space or space-time.

2.4 Spinors, Majorana and Weyl conditions

The final paragraph on abstract Clifford algebras is dedicated to the objects that actually transform under Clifford elements. Since the complex Pauli matrices (and their tensor products) give rise to representations of Clifford algebras it is natural to define spinors as complex valued vectors.

Definition 3. *The vector space of complex n-spinors is*

$$\Delta_n := \mathbb{C}^{2^k} = \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2 \qquad \text{for} \quad n = 2k, 2k+1.$$

Elements of Δ_n are called complex spinors or Dirac spinors.

Some properties of spinors can be derived from simple considerations on Clifford products of basis elements. Let us for example consider the case of an even dimensional vector space $\mathbb{R}^{2k}$. In this case the product of all basis elements

$$\Gamma_{2k+1} = \pm a\, \Gamma_1 \cdot \ldots \cdot \Gamma_d \tag{2.45}$$

(the name Γ_{2k+1} is just convention and originates from 4 dimensions where the product of all Γ-matrices was usually called Γ_5) results in an element which is not proportional to $\mathbb{1}$, and generally is linearly independent of all other basis elements and their products. This is a consequence of the above equation (2.37) *2. Taking the square of any product of basis elements in the Clifford algebra will result in an element proportional to one. Depending on the particular case, we introduce an imaginary unit into the product of all basis elements to ensure that it squares to $+\,\mathbb{1}$ (hence $a = \{1, i\}$). Since it squares to $\mathbb{1}$ while not representing a basis vector, this element can be taken as an involution. Being a product of an even number of basis element, it will also commute with all other products containing an even number of basis vectors, as for example the elements of $SPIN(2k)$. This involution can therefore be used to split the vector space (i.e. the space of Dirac spinors) into two parts: those with positive, respectively negative eigenvalue under Γ_{2k+1}. Explicitly: take a spinor λ and measure its eigenvalue under Γ^{2k+1}. Since Lorentz generators commute with Γ^{2k+1}, this eigenvalue will not change under Lorentz rotations and hence serves as an invariant which is called handedness or chirality. It therefore makes sense to define

*2 Compare the specific example (2.39).

Definition 4. *The elements of the sub-spaces with positive or negative eigenvalue under* Γ_{2k+1} *are called (positive or negative) Weyl-spinors.*

Weyl-spinors are elements of irreducible modules, i.e. sub vector spaces that do not mix under the action of the spin group. By construction it is clear that Weyl-spinors exist in all even dimensions and never in odd dimensions.
Before describing another involution which can be used to find irreducible sub-modules, we need to state a fundamental theorem of Clifford algebras without proving it. Its content is simple. It merely states the fact that two different representations of a Clifford algebra over the same base space are isomorphic and hence related by a similarity transformation. The proof is not difficult and the isomorphism can be constructed explicitly, however, in order to keep the pace we refer the interested reader to [27]. We are now ready for the following considerations.

It can be seen from (2.37) that despite considering real vector spaces $\mathbb{R}^d$, the representation of the Clifford algebra may involve imaginary units and thus acts on complex-valued spinors (as laid out in Definition 3). It is easy to see, however, that if a set of Γ-matrices fulfils the Clifford relations (2.26) then so does the set Γ^*, i.e. the complex conjugate. In other words both are representation over the same quadratic form. Due to the fundamental Lemma just mentioned, they must be isomorphic. Therefore a matrix B exists such that

$$\Gamma^*_\mu = B\,\Gamma_\mu\,B^{-1}, \qquad \text{and thus also} \qquad \Gamma^*_{\mu\nu} = B\,\Gamma_{\mu\nu}\,B^{-1}. \tag{2.46}$$

Since we have introduced the notion of complex conjugation on the Clifford algebra, one can now pose the question whether it might be possible to find purely real-valued representations of its elements. One way is to just go and try. But there is a more elegant way leading to **Majorana** spinors. These are defined through the observation that also Γ^T (transposed matrices) furnish a representation if the original Γ's do. Again the same argument as above applies, and there exists a matrix C such that

$$\Gamma^T_\mu = -C\,\Gamma_\mu\,C^{-1}, \qquad \text{and thus also} \qquad \Gamma^T_{\mu\nu} = -C\,\Gamma_{\mu\nu}\,C^{-1} \tag{2.47}$$

where we have introduced a minus sign which physically allows us to identify the matrix C with the charge conjugation matrix. Next, using the fact that a spinor transforms under a $SPIN$-transformation as (additional factor one half in comparison to above formula (2.40) because of anti-symmetry in the indices)

$$\delta\lambda = \frac{1}{4}\,\omega^{mn}\,\Gamma_{mn}\,\lambda$$

we calculate

$$\delta(B^{-1}\lambda^*) = B^{-1}\delta\lambda^* = B^{-1}\left(\frac{1}{4}\,\omega^{mn}(\Gamma_{mn}\lambda)^*\right) = \frac{1}{4}\,\omega^{mn}B^{-1}B\Gamma_{mn}B^{-1}\lambda^* = \frac{1}{4}\,\omega^{mn}\Gamma_{mn}(B^{-1}\lambda^*). \tag{2.48}$$

This equation means that also $B^{-1}\lambda^*$ transforms under $SPIN(d)$ in the same way that λ does. Hence,

it might be the case the the complex conjugate of a spinor can just be calculated by $\lambda^* = B\lambda$. This can then be understood as some kind of reality condition of the spinor.

Using the fundamental lemma once more there must be a relation between the matrices B and C, since both representations must also be isomorphic. The precise relation between these matrices depends crucially one the signature of the vector space $\mathbb{R}^{p,q}$ under consideration.

Having argued that the matrix B might relate the spinor λ to its complex conjugate spinor $\lambda^* = B\,\lambda$, we take the complex conjugate of this equation and plug it into itself to find

$$\lambda \;=\; B^* B\,\lambda \;=\; \epsilon\,\lambda\,,$$

since B^*B commutes with the representation when taking the complex conjugate twice (2.46) as can be seen from the following computation:

$$\Gamma_m \;=\; (\Gamma^*_m)^* \;=\; B^*B\,\Gamma_m\,B^{-1}B^{-1*}.$$

By Schur's Lemma B^*B will thus have to be a multiple of the unit matrix $B^*B = \epsilon\; Id$. One can work out the precise value for ϵ in various space-time dimensions and signatures. Below we will give a table for the specific case of one time dimension and $d-1$ space dimensions. Majorana spinors then exist if $\epsilon = 1$ (since then B becomes an involution) which can be shown to be the case if $d = 2, 4 \bmod 8$.

Finally, one could ask in which dimensions -assuming 'physical' signature $\mathbb{R}^{1,d-1}$- the Majorana condition is compatible with the condition for Weyl spinors. This merely depends on whether or not we had to introduce the imaginary unit into Γ^{2k+1} (2.45) to ensure its squaring to $\mathbb{1}$ [*3]. It turns out that for $d = 2 \bmod 4$ no imaginary unit is required. Hence is the Majorana condition compatible with the Weyl condition in $d = 2$, 10, 18, 26, ... In the table below, we have indicated the reducibility of the spinors in physical signature up to eleven dimensions.

The first case, $d = 2$, happens to be the dimension of the world-sheet of a string, the second case, $d = 10$, is the dimensionality of super string theories, and the case $d = 26$ is the critical dimension of bosonic string theory.

d	2	3	4	5	6	7	8	9	10	11
	M-W	M	M/W	-	W	-	W	M	M-W	M.

M stands for Majorana, W for Weyl, M/W stands for either Majorana or Weyl, M-W for Majorana and Weyl, correspondingly a dash means neither M nor W.

[*3] In the physical signature example underneath (2.36), we had to introduce the imaginary unit. You can check by hand that you would also have to introduce it when defining Γ^9 in $\mathbb{R}^{1,7}$. You can also check explicitly that you don't need to introduce the imaginary unit for Γ^3 in $\mathbb{R}^{1,1}$ or Γ^7 in $\mathbb{R}^{1,5}$ or Γ^{11} in $\mathbb{R}^{1,9}$.

Summary 2. *This finishes the overview into Clifford algebras. From now on we will present explicit calculations. After having understood the basic ideas behind Clifford algebras, the reader might be tempted to assume that calculations are easy to do. This will turn out to be a false assumption since the representation of* $\mathrm{Cliff}(\mathbb{R}^{1,10})$ *is in terms of 32-by-32 matrices. The authors therefore thought to present some explicit calculations in order to see how they are done in general.*
Unfortunately, many calculations are depending very much on conventions. The first choice effecting the representation of the Clifford algebras is of course whether mostly plus or mostly minus quadratic forms are used. Secondly, we have given an explicit representation of basis elements in (2.37) where others are possible but of course isomorphic (due to the fundamental theorem on Clifford algebras). Multiplying purely real Γ_i *matrices by the imaginary unit results in purely imaginary basis elements. Since eleven dimensions indeed do admit a Majorana condition imposed on the spinors, some papers talk of purely imaginary, others of purely real representations of Clifford algebras.*

2.5 Eleven dimensional Clifford Algebra

In subsection 2.2 we have already constructed the eleven dimensional Euclidean Clifford algebra (2.38). In eleven dimensional SUGRA we also need a realisation of an eleven dimensional Clifford algebra of a pseudo Euclidean space with Minkowskian signature. The fundamental anticommutation relation of the corresponding Clifford algebra reads

$$\{\,\Gamma^a,\Gamma^b\,\} \quad = \quad 2\,\eta^{ab}\,\mathbb{1}_{32}, \qquad\qquad \eta^{ab} \;=\; \mathrm{diag}(-1,1,\ldots,1) \tag{2.49}$$

A representation by real matrices Γ^a can be obtained by taking appropriate tensor products of the Pauli matrices (2.10) and $\epsilon = i\tau_2$:

$$\begin{aligned}
\Gamma^0 &= -\epsilon\otimes\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\tau_3 & \Gamma^1 &= \epsilon\otimes\epsilon\otimes\epsilon\otimes\epsilon\otimes\tau_1\\
\Gamma^2 &= \epsilon\otimes\mathbb{1}\otimes\tau_1\otimes\epsilon\otimes\tau_1 & \Gamma^3 &= \epsilon\otimes\mathbb{1}\otimes\tau_3\otimes\epsilon\otimes\tau_1\\
\Gamma^4 &= \epsilon\otimes\tau_1\otimes\epsilon\otimes\mathbb{1}\otimes\tau_1 & \Gamma^5 &= \epsilon\otimes\tau_3\otimes\epsilon\otimes\mathbb{1}\otimes\tau_1\\
\Gamma^6 &= \epsilon\otimes\epsilon\otimes\mathbb{1}\otimes\tau_1\otimes\tau_1 & \Gamma^7 &= \epsilon\otimes\epsilon\otimes\mathbb{1}\otimes\tau_3\otimes\tau_1\\
\Gamma^8 &= \epsilon\otimes\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\epsilon & \Gamma^9 &= \tau_1\otimes\mathbb{1}_{16}\\
\Gamma^{10} &= \tau_3\otimes\mathbb{1}_{16}
\end{aligned} \tag{2.50}$$

It turns out that the algebra generated by the eleven gamma matrices given above and all antisymmetrised products of them is isomorphic to the space of real 32×32 matrices, i.e.

$$\mathrm{Cliff}(\mathbb{R}^{1,10}) \quad\equiv\quad Mat(32,\mathbb{R})\oplus Mat(32,\mathbb{R})\ . \tag{2.51}$$

In supergravity we are concerned with spinorial representations only. As discussed in a previous sections this restricts the consideration to the even dimensional Clifford algebra or picking up one of the

two $Mat(32,\mathbb{R})$. In this sense we sometimes claim $\text{Cliff}(\mathbb{R}^{1,10}) \,=\, Mat(32,\mathbb{R})$.

In the following we work in this representation of the Γ-matrices. For most computations apart from those in section 7 the details of the representation are insignificant. But of utmost importance in this note are the following relations

$$(\Gamma^a)^\dagger \,=\, \Gamma^0\,\Gamma^a\,\Gamma^0 \qquad \Leftrightarrow \qquad (\Gamma^a)^T \,=\, \Gamma^0\,\Gamma^a\,\Gamma^0 \tag{2.52}$$

Proposition 1. *All antisymmetric products of Γ-matrices not equal to $\mathbb{1}$ are traceless.*

Proof. For $\Gamma^{a_1..a_{2i}}$ one has

$$\begin{aligned} Tr(\,\Gamma^{a_1..a_{2i}}\,) &= Tr(\,\Gamma^{a_1}\cdot\ldots\cdot\Gamma^{a_{2i}}\,) \\ &= Tr(\,\Gamma^{a_2}\cdot\ldots\cdot\Gamma^{a_{2i}}\,\Gamma^{a_1}\,) \\ &= (-1)^{2i-1}\,Tr(\,\Gamma^{a_1..a_{2i}}\,) \end{aligned}$$

For $\Gamma^{a_1..a_{2i+1}}$ with $2i+1 \,<\, 11$ there exists an $a_j \notin (a_1,\ldots,a_{2i+1})$

$$\begin{aligned} Tr(\,\Gamma^{a_1..a_{2i+1}}\,) &= Tr(\,(\Gamma^{a_j}\Gamma_{a_j})\,\Gamma^{a_1}\cdot\ldots\cdot\Gamma^{a_{2i+1}}\,) \\ &= Tr(\,\Gamma_{a_j}\,\Gamma^{a_1}\cdot\ldots\cdot\Gamma^{a_{2i+1}}\,\Gamma^{a_j}\,) \\ &= (-1)^{2i+1}\,Tr(\,\Gamma^{a_1..a_{2i+1}}\,) \end{aligned}$$

□

Proposition 2. *We note the following useful identity without proof:*

$$\Gamma^{a_j\ldots a_1}\Gamma_{b_1\ldots b_k} \;=\; \sum_{l=0}^{\min(j,k)} l!\begin{pmatrix} j \\ l \end{pmatrix}\begin{pmatrix} k \\ l \end{pmatrix}\,\delta^{[a_1}_{[b_1}\cdots\delta^{a_l}_{b_l}\,{\Gamma^{a_j\ldots a_{l+1}]}}_{b_{l+1}\ldots b_k]}\;. \tag{2.53}$$

Proposition 3. *The following identity holds*

$$\frac{1}{32}\,Tr\,(\,\Gamma^{a_n\ldots a_1}\Gamma_{b_1\ldots b_n}\,) \;=\; \delta^{a_1\ldots a_n}_{b_1\ldots b_n} \tag{2.54}$$

Proof. The proof is a direct consequence of computing the trace of the identity (2.53) combined with the tracelessness of antisymmetrised Γ-matrices.

□

Definition 5. *With* $\psi_\mu = \psi_\mu^\alpha$ *a Majorana spinor 1-form is given by*

$$\psi = \psi_\mu \, dx^\mu \ .$$

Definition 6. *Dirac/Majorana conjugate*

$$\bar{\psi}^D = \psi^\dagger A \tag{2.55}$$
$$\bar{\psi}^M = \psi^T C \tag{2.56}$$

Here A and C are intertwiners of representations of the Γ-algebra[*4]

$$(\Gamma^a)^\dagger = A \, \Gamma^a \, A \qquad \Leftrightarrow \qquad (\Gamma^a)^T = C \, \Gamma^a \, C$$

Comparison with eq. (2.52) leads to $A = \Gamma^0$ and $C = \Gamma^0$.

Remark 1. Exchanging two spinors produces a sign (due to the Fermi statistics of fermions).

Proposition 4.

$$\bar{\psi} \, \Gamma^{a_1 \dots a_i} \wedge \psi = 0 \ , \qquad i = 0, 3, 4, 7, 8, 11 \tag{2.57}$$

Proof. Write eq. (2.57) in coordinates, i.e.

$$\bar{\psi} \, \Gamma^{a_1 \dots a_i} \wedge \psi = \left(\bar{\psi}_{[\mu} \, \Gamma^{a_1 \dots a_i} \, \psi_{\nu]} \right) dx^\mu \wedge dx^\nu$$

Now performing a transposition in the spin-space, the "scalar" $\bar{\psi} \, \Gamma^{a_1 \dots a_i} \wedge \psi$ does not transform and we obtain using eq. (2.52) and the anticommuting property of spinors:

$$\begin{aligned}
\bar{\psi}_{[\mu} \, \Gamma^{a_1 \dots a_i} \, \psi_{\nu]} &\overset{!}{=} \left(\bar{\psi}_{[\mu} \, \Gamma^{a_1 \dots a_i} \, \psi_{\nu]} \right)^T \\
&= -\, \psi^T_{[\nu} \, (\Gamma^{a_i})^T \, \dots \, (\Gamma^{a_1})^T \, (\Gamma^0)^T \psi_{\mu]} \\
&= (-1)^i \, \bar{\psi}_{[\nu} \, \Gamma^{a_i \dots a_1} \, \psi_{\mu]} \\
&= -\, (-1)^{\frac{i(i+1)}{2}} \, \bar{\psi}_{[\mu} \, \Gamma^{a_1 \dots a_i} \, \psi_{\nu]}
\end{aligned} \tag{2.58}$$

The value of the sign on the right hand side of the last equation is depicted in the following table:

[*4] This is consistent with (2.47) due to $(\Gamma^0)^{-1} = -\Gamma^0$.

i	$-(-1)^{\frac{i(i+1)}{2}}$	$\bar{\psi}\,\Gamma^{a_1\,\dots\,a_i}\wedge\psi$
0	-1	0
1	+1	
2	+1	
3	-1	0
4	-1	0
5	+1	
6	+1	
7	-1	0
8	-1	0
9	+1	
10	+1	
11	-1	0

Due to the type of signs the terms pick up during the manipulation some of them can be seen to vanish identically. In fact, up to duality, just the terms with one, two or five Gamma matrices persist.

□

Proposition 5. *A consequence is the elementary Fierz identity:*

$$\psi)\wedge(\bar{\psi} = \frac{1}{32}\left(\Gamma^{c_1}\,(\bar{\psi}\,\Gamma_{c_1}\,\wedge\,\psi) + \frac{\Gamma^{c_2 c_1}}{2!}\,(\bar{\psi}\,\Gamma_{c_1 c_2}\,\wedge\,\psi) + \frac{\Gamma^{c_5\dots c_1}}{5!}\,(\bar{\psi}\,\Gamma_{c_1\dots c_5}\,\wedge\,\psi)\right)\,. \tag{2.59}$$

Proof. According to (2.51) a generic real 32×32-matrix Γ can be expanded in a basis of the Clifford algebra as

$$\Gamma \;=\; \sum_{i=0}^{5} C_{|i|}\cdot\Gamma^{|i|} \tag{2.60}$$

with $\Gamma^{|i|} = \Gamma^{a_1\dots a_i}$. Using the identity (2.54) one obtains

$$\begin{aligned} Tr(\Gamma\,\Gamma_{|k|}) &= \sum_{i=0}^{5} C_{|i|}\cdot Tr(\Gamma^{|i|}\,\Gamma_{|k|}) \\ &= \sum_{i=0}^{5} C_{|i|}\;32\,(-1)^{\frac{i(i-1)}{2}}\,\delta^{|i|}{}_{|k|} \\ &= k!\cdot C_{|k|}\;32\,(-1)^{\frac{k(k-1)}{2}} \end{aligned}$$

or

$$C_{|k|} \;=\; \frac{(-1)^{\frac{k(k-1)}{2}}}{32\cdot k!}\, Tr(\Gamma\,\Gamma_{|k|})$$

This leads to

$$\Gamma \;=\; \frac{1}{32}\sum_{i=0}^{5}\frac{(-1)^{\frac{i(i-1)}{2}}}{i!}\, Tr(\Gamma\,\Gamma_{|i|})\cdot\Gamma^{|i|} \tag{2.61}$$

Choosing $(\Gamma)^{\alpha\beta} \;=\; \psi^{\alpha}_{[\mu})\,(\bar\psi^{\beta}_{\nu]}$, i.e. the left hand side of eq. (2.59) one obtains

$$\begin{aligned}\psi^{\alpha}_{[\mu})\,(\bar\psi^{\beta}_{\nu]} &= \frac{1}{32}\sum_{i=0}^{5}\frac{(-1)^{\frac{i(i-1)}{2}}}{i!}\,\underbrace{Tr(\psi^{\gamma}_{[\mu})\,(\bar\psi^{\delta}_{\nu]}\,(\Gamma_{|i|})_{\delta\gamma})}_{(\bar\psi_{[\mu}\,\Gamma_{|i|}\,\psi_{\nu]})}\cdot\Gamma^{|i|}\\ &= \frac{1}{32}\sum_{i=0}^{5}\frac{(-1)^{\frac{i(i-1)}{2}}}{i!}\,(\bar\psi_{[\mu}\,\Gamma_{|i|}\psi_{\nu]})\cdot\Gamma^{|i|}\end{aligned}$$

Due to eq. (2.57) only three terms can contribute to the sum. Absorbing the signs in the reordering of the indices of the Gamma-matrices one finally obtains

$$\psi^{\alpha}_{[\mu})\,(\bar\psi^{\beta}_{\nu]} \;=\; \frac{1}{32}\left((\bar\psi_{[\mu}\,\Gamma_{c_1}\psi_{\nu]})\cdot\Gamma^{c_1} \;+\; (\bar\psi_{[\mu}\,\Gamma_{c_1c_2}\psi_{\nu]})\cdot\frac{\Gamma^{c_2c_1}}{2!} \;+\; (\bar\psi_{[\mu}\,\Gamma_{c_1\ldots c_5}\psi_{\nu]})\cdot\frac{\Gamma^{c_5\ldots c_1}}{5!}\right) \tag{2.62}$$

which is the expression (2.59) in coordinates.

□

Definition 7. *Open/Closed Terms: Terms of the type $\bar\psi_{[\mu}\,\Gamma^{(j)}\,\psi_{\nu]}$, i.e. with all spinor indices contracted, are called closed terms. Opposite to that we call terms of the type $\Gamma^{(i)}\psi_{[\mu}\,\bar\psi_{\nu]}\Gamma^{(j)}$ open.*

It is the nature of Fierz identities to express open terms in terms of closed terms and vice versa.

Proposition 6. *The duality relation of the 11d Clifford algebra reads*[*5]

$$\Gamma^{a_1\ldots a_p} \;=\; \frac{(-1)^{\frac{(p+1)(p-2)}{2}}}{(11-p)!}\,\varepsilon^{a_1\ldots a_p a_{p+1}\ldots a_{11}}\,\Gamma_{a_{p+1}\ldots a_{11}}\;. \tag{2.63}$$

[*5] From (2.50) it follows $\Gamma^{0\ldots10} \;=\; -\mathbb{1}_{32}$ and correspondingly we define $\varepsilon^{0\ldots10} \;=\; -1$.

Proof. We start again from (2.60) and chose $\Gamma = \Gamma^{a_1 \ldots a_p}$

$$\Gamma^{a_1 \ldots a_p} = \sum_{i=0}^{5} C^{|i|} \cdot \Gamma_{|i|} \ . \tag{2.64}$$

Now we multiply by $\Gamma^{|k|}$ with the length $|k| = 11 - p$. We obtain

$$\Gamma^{a_1 \ldots a_p} \Gamma^{a_{p+1} \ldots a_{11}} = \sum_{i=0}^{5} C^{|i|} \cdot \Gamma_{|i|} \Gamma^{|k|} \ . \tag{2.65}$$

If all indices are different, then due to (2.50) we have $\Gamma^{a_1 \ldots a_p a_{p+1} \ldots a_{11}} = -\mathbb{1}_{32}$ and after taking the trace the left hand side of (2.65) reads (cf. footnote on page 24):

$$Tr\,(\,\Gamma^{a_1 \ldots a_p} \Gamma^{a_{p+1} \ldots a_{11}}\,) = 32\,\varepsilon^{a_1 \ldots a_{11}} \ , \tag{2.66}$$

while the right hand side reads

$$Tr(\sum_{i=0}^{5} C^{|i|} \cdot \Gamma_{|i|} \Gamma^{|k|}\,) = 32\, C^{b_{11} \ldots b_{p+1}}\, \delta^{a_{p+1} \ldots a_{11}}_{b_{p+1} \ldots b_{11}} = 32\,(11-p)!\, C^{a_{11} \ldots a_{p+1}} \ . \tag{2.67}$$

Comparing both results with each other one obtains

$$C^{a_{11} \ldots a_{p+1}} = \frac{1}{(11-p)!}\, \varepsilon^{a_1 \ldots a_{11}} \ , \tag{2.68}$$

and finally

$$\Gamma^{a_1 \ldots a_p} = \frac{1}{(11-p)!}\, \varepsilon^{a_1 \ldots a_{11}}\, \Gamma_{a_{11} \ldots a_{p+1}} \tag{2.69}$$

Ordering the indices produces a sign $(-1)^{\frac{(11-p-1)(11-p)}{2}}$, which using modulo 4 arithmetic can be written as

$$(-1)^{\frac{(11-p-1)(11-p)}{2}} = (-1)^{\frac{(p+1)(p-2)}{2}} \ . \tag{2.70}$$

□

Homework:

Exercise 1. *Prove in analogy to (2.58) the identity. Here* ϵ *and* ψ *commute and* $\Gamma^{(j)} = \Gamma^{a_1 \ldots a_j}$*:*

$$\bar{\psi}_\mu\, \Gamma^{(j)}\, \epsilon = -\,(-1)^{\frac{j(j+1)}{2}}\, \bar{\epsilon}\, \Gamma^{(j)}\, \psi_\mu \ . \tag{2.71}$$

2.6 Cremmer-Julia-Scherk Fierz Identity

In the application of the eleven dimensional Clifford algebra to supergravity the efficient handling of Fierz identities becomes essential. The topic is technically involved. A good reference for a deeper study is [28]. Another source of information is [29]. In this section we want to prove the most important Fierz identity appearing in computations within eleven dimensional supergravity and providing a taste of the technicalities showing up in this context. It is dubbed the CJS Fierz identity and given below:

$$
\begin{aligned}
&\frac{1}{8}\,\Gamma^{\mu\nu\alpha\beta\gamma\delta}\,\psi_\nu\bar{\psi}_\alpha\,\Gamma_{\beta\gamma} \;-\; \frac{1}{8}\,\Gamma_{\beta\gamma}\,\psi_\nu\bar{\psi}_\alpha\,\Gamma^{\mu\nu\alpha\beta\gamma\delta} \\
&\quad -\; \frac{1}{4}\,\Gamma^{\mu\nu\alpha\beta\delta}\psi_\nu\bar{\psi}_\alpha\Gamma_\beta \;+\; \frac{1}{4}\,\Gamma_\beta\psi_\nu\bar{\psi}_\alpha\Gamma^{\mu\nu\alpha\beta\delta} \\
&-\; 2\,g^{\beta[\alpha}\Gamma^{\delta\mu\nu]}\psi_\nu\bar{\psi}_\alpha\Gamma_\beta \;-\; 2\,\Gamma_\beta\psi_\nu\bar{\psi}_\alpha g^{\beta[\alpha}\Gamma^{\delta\mu\nu]} \\
&\qquad +\; 2\,g^{\beta[\alpha}\Gamma^{\delta\mu\nu]}\,\bar{\psi}_\alpha\Gamma_\beta\psi_\nu \;=\; 0
\end{aligned}
\tag{2.72}
$$

Proof. Inserting the basic Fierz identity eq. (2.62) into eq. (2.72) and ordering after closed terms one obtains:

$$
\begin{aligned}
\text{Eq. (2.72)} \;=\;& \frac{1}{32}\left\{\;1^{\text{st}}-\text{term}\;\right\}\psi_{[\nu}\,\Gamma_{|c_1|}\,\psi_{\alpha]} \;+\; \frac{1}{32}\left\{\;2^{\text{nd}}-\text{term}\;\right\}\frac{1}{2!}\,\psi_{[\nu}\,\Gamma_{|c_1c_2|}\,\psi_{\alpha]} \\
+\;& \frac{1}{32}\left\{\;3^{\text{th}}-\text{term}\;\right\}\frac{1}{5!}\,\psi_{[\nu}\,\Gamma_{|c_1\ldots c_5|}\,\psi_{\alpha]}
\end{aligned}
$$

Each of the three terms must vanish separately. There is a small difference in the treatment of the first and the last two terms. This is due to the presence of closed term $2\,g^{\beta[\alpha}\Gamma^{\delta\mu\nu]}\,\bar{\psi}_\alpha\Gamma_\beta\psi_\nu$ in eq. (2.72), which contributes only to the first but not to the last two terms. The part which is common to all three terms and which must be rewritten by Fierzing consists of the first six terms in eq. (2.72) and we are going to rewrite it now. A unified notation allowing to handle the common part (c.p.) of all three terms at once is ($\Gamma^{(j)}$ denotes a totally antisymmetric matrix with j indices)

$$
\begin{aligned}
\left\{\;j^{\text{th}}-\text{term},\;\text{c.p.}\;\right\} \;=\;& \underbrace{\frac{1}{8}\,\Gamma^{\mu\nu\alpha\beta\gamma\delta}\,\Gamma^{(j)}\,\Gamma_{\beta\gamma} \;-\; \frac{1}{8}\,\Gamma_{\beta\gamma}\,\Gamma^{(j)}\,\Gamma^{\mu\nu\alpha\beta\gamma\delta}}_{(1)} \\
& \underbrace{-\,\frac{1}{4}\,\Gamma^{\mu\nu\alpha\beta\delta}\,\Gamma^{(j)}\,\Gamma_\beta \;+\; \frac{1}{4}\,\Gamma_\beta\,\Gamma^{(j)}\,\Gamma^{\mu\nu\alpha\beta\delta}}_{(2)} \\
& \underbrace{-\,2\,g^{\beta[\alpha}\,\Gamma^{\delta\mu\nu]}\,\Gamma^{(j)}\,\Gamma_\beta \;-\; 2\,\Gamma_\beta\,\Gamma^{(j)}\,g^{\beta[\alpha}\,\Gamma^{\delta\mu\nu]}}_{(3)}\;.
\end{aligned}
\tag{2.73}
$$

To simplify the terms on the right hand side we need two identities of Clifford matrices, which are

easy to prove. The first by direct computation and the second by iterating the first one twice:

$$\Gamma_\beta\,\Gamma_{a_1\ldots a_j}\,\Gamma^\beta \quad = \quad (-1)^j\,(D-2\,j)\,\Gamma_{a_1\ldots a_j} \tag{2.74}$$

$$\Gamma_{\beta\gamma}\,\Gamma_{a_1\ldots a_j}\,\Gamma^{\beta\gamma} \quad = \quad -\left[\,(D-2\,j)^2\,-\,D\,\right]\,\Gamma_{a_1\ldots a_j}\ . \tag{2.75}$$

Proof. of (2.75)

$$\begin{aligned}\Gamma_{\beta\gamma}\,\Gamma_{a_1\ldots a_j}\,\Gamma^{\beta\gamma} &= -\,(\,\Gamma_\beta\Gamma_\gamma-\eta_{\beta\gamma}\,)\,\Gamma_{a_1\ldots a_j}\,\left(\Gamma^\gamma\Gamma^\beta-\eta^{\gamma\beta}\right)\\ &= -\,(D-2\,j)^2\,\Gamma_{a_1\ldots a_j}\,+\,D\,\Gamma_{a_1\ldots a_j}\,+\,D\,\Gamma_{a_1\ldots a_j}\,-\,D\,\Gamma_{a_1\ldots a_j}\\ &= -\left[\,(D-2\,j)^2\,-\,D\,\right]\,\Gamma_{a_1\ldots a_j}\end{aligned}$$

□

In eleven dimensional SUGRA we must specify $D\,=\,11$, obviously.

Ad (1) :

Instead of (1) we consider

$$\frac{1}{8}\,\Gamma^{\mu\nu\alpha\delta}\Gamma_{\beta\gamma}\,\Gamma^{(j)}\,\Gamma^{\beta\gamma}\,-\,\frac{1}{8}\,\Gamma^{\beta\gamma}\,\Gamma^{(j)}\,\Gamma_{\beta\gamma}\Gamma^{\mu\nu\alpha\delta} \tag{2.76}$$

which using eq. (2.75) reads

$$(2.76) \quad = \quad -\frac{1}{8}\,\left[\,(D-2\,j)^2-D\,\right]\,\left[\Gamma^{\mu\nu\alpha\delta},\,\Gamma^{(j)}\right]\ . \tag{2.77}$$

Alternatively, using (2.53), it can be rewritten as

$$\begin{aligned}(2.76) \quad &= \quad \frac{1}{8}\,\left\{\Gamma^{\mu\nu\alpha\delta}{}_{\beta\gamma}\,+\,8\,\delta^{[\delta}_{[\beta}\Gamma^{\mu\nu\alpha]}{}_{\gamma]}\,+\,2\cdot 6\cdot\delta^{[\delta}_{[\beta}\delta^{\alpha}_{\gamma]}\Gamma^{\mu\nu]}\right\}\,\Gamma^{(j)}\,\Gamma^{\beta\gamma}\\ &- \quad \frac{1}{8}\,\Gamma^{\beta\gamma}\,\Gamma^{(j)}\,\left\{\Gamma_{\beta\gamma}{}^{\mu\nu\alpha\delta}\,+\,8\,\delta^{[\mu}_{[\gamma}\Gamma_{\beta]}{}^{\nu\alpha\delta]}\,+\,2\cdot 6\cdot\delta^{[\mu}_{[\gamma}\delta^{\nu}_{\beta]}\Gamma^{\alpha\delta]}\right\}\end{aligned} \tag{2.78}$$

Comparing (2.77) with (2.78) one obtains

$$\begin{aligned}\frac{1}{8}\,\Gamma^{\mu\nu\alpha\delta}{}_{\beta\gamma}\,\Gamma^{(j)}\,\Gamma^{\beta\gamma}\,-\,\frac{1}{8}\,\Gamma^{\beta\gamma}\,\Gamma^{(j)}\,\Gamma^{\mu\nu\alpha\delta}{}_{\beta\gamma} \quad &= \quad -\frac{1}{8}\,\left[\,(D-2\,j)^2-D\,\right]\,\left[\Gamma^{\mu\nu\alpha\delta},\,\Gamma^{(j)}\right]\\ &\quad -\frac{1}{8}\,\left\{8\,\delta^{[\delta}_{[\beta}\Gamma^{\mu\nu\alpha]}{}_{\gamma]}\,+\,2\cdot 6\cdot\delta^{[\delta}_{[\beta}\delta^{\alpha}_{\gamma]}\Gamma^{\mu\nu]}\right\}\,\Gamma^{(j)}\,\Gamma^{\beta\gamma}\\ &\quad +\frac{1}{8}\,\Gamma^{\beta\gamma}\,\Gamma^{(j)}\,\left\{8\,\delta^{[\mu}_{[\gamma}\Gamma_{\beta]}{}^{\nu\alpha\delta]}\,+\,2\cdot 6\cdot\delta^{[\mu}_{[\gamma}\delta^{\nu}_{\beta]}\Gamma^{\alpha\delta]}\right\}\end{aligned}$$

The left hand side is exactly what we denoted by (1) in (2.73). The two terms with the factors "$2\cdot 6$" cancel each other and one obtains

$$
\begin{aligned}
(1) \quad = \quad & -\frac{1}{8}\,\big[\,(D-2\,j)^2 - D\,\big]\,\Big[\,\Gamma^{\mu\nu\alpha\delta},\,\Gamma^{(j)}\,\Big] \\
& -\underbrace{\delta^{[\delta}_{[\beta}\Gamma^{\mu\nu\alpha]}{}_{\gamma]}\,\Gamma^{(j)}\,\Gamma^{\beta\gamma}}_{\Gamma^{[\mu\nu\alpha|}{}_{\gamma}\,\Gamma^{(j)}\,\Gamma^{|\delta]\gamma}} + \underbrace{\Gamma^{\beta\gamma}\,\Gamma^{(j)}\,\delta^{[\mu}_{[\gamma}\Gamma_{\beta]}{}^{\nu\alpha\delta]}}_{\Gamma^{\gamma[\mu|}\,\Gamma^{(j)}\,\Gamma_{\gamma}{}^{|\nu\alpha\delta]}}
\end{aligned}
\tag{2.79}
$$

To get rid of the contractions over γ in the last two terms we obtain by (2.74)

$$
\begin{aligned}
-\,\Gamma^{[\mu\nu\alpha|}{}_{\gamma}\,\Gamma^{(j)}\,\Gamma^{|\delta]\gamma} \quad &= \quad \Big(\Gamma^{[\mu\nu\alpha|}\,\Gamma_{\gamma} \,-\, 3\,\delta^{[\alpha}_{\gamma}\Gamma^{\mu\nu}\Big)\,\Gamma^{(j)}\,\Big(\Gamma^{\gamma}\Gamma^{|\delta]} \,-\, \eta^{\gamma|\delta]}\Big) \\
&= \quad (-1)^j\,(D-2\,j)\,\Gamma^{[\mu\nu\alpha|}\,\Gamma^{(j)}\,\Gamma^{|\delta]} \,-\, \Gamma^{\mu\nu\alpha\delta}\,\Gamma^{(j)} \,-\, 3\,\delta^{[\alpha}_{\gamma}\Gamma^{\mu\nu}\,\Gamma^{(j)}\,\Gamma^{\gamma}\Gamma^{|\delta]} \,+\, 0
\end{aligned}
$$

and analogously

$$
\Gamma^{\gamma[\mu|}\,\Gamma^{(j)}\,\Gamma_{\gamma}{}^{|\nu\alpha\delta]} \quad = \quad -\,(-1)^j\,(D-2\,j)\,\Gamma^{[\mu|}\,\Gamma^{(j)}\,\Gamma^{|\nu\alpha\delta]} \,+\, 3\,\Gamma^{[\mu|}\,\Gamma^{\gamma}\,\Gamma^{(j)}\,\delta^{|\nu}_{\gamma}\Gamma^{\alpha\delta]} \,+\, \Gamma^{(j)}\,\Gamma^{\mu\nu\alpha\delta}\;.
$$

Plugging the last two results back into (2.79) the two terms with the factor "3" cancel each other and one obtains

$$
\begin{aligned}
(1) \quad = \quad & -\frac{1}{8}\,\big[\,(D-2\,j)^2 - D + 8\,\big]\,\Big[\,\Gamma^{\mu\nu\alpha\delta},\,\Gamma^{(j)}\,\Big] \\
& + (-1)^j\,(D-2\,j)\,\Big\{\,\Gamma^{[\mu\nu\alpha|}\,\Gamma^{(j)}\,\Gamma^{|\delta]} \,-\, \Gamma^{[\mu|}\,\Gamma^{(j)}\,\Gamma^{|\nu\alpha\delta]}\Big\}
\end{aligned}
\tag{2.80}
$$

It is useful to observe, that the expression in curly brackets has the same tensor structure as (3) in (2.73).

Ad (2) :

In order to simplify the terms in (2) we proceed similarly to the previous calculations:

$$
\begin{aligned}
(2) \quad &= \quad -\frac{1}{4}\,\Gamma^{\mu\nu\alpha}{}_{\beta}{}^{\delta}\,\Gamma^{(j)}\,\Gamma^{\beta} \,+\, \frac{1}{4}\,\Gamma^{\beta}\,\Gamma^{(j)}\,\Gamma^{\mu\nu\alpha}{}_{\beta}{}^{\delta} \\
&= \quad \frac{1}{4}\,\Big\{\,\Gamma^{\mu\nu\alpha\delta}\,\Gamma_{\beta} \,-4\,\delta^{[\delta}_{\beta}\Gamma^{\mu\nu\alpha]}\,\Big\}\,\Gamma^{(j)}\,\Gamma^{\beta} \,-\, \frac{1}{4}\,\Gamma^{\beta}\,\Gamma^{(j)}\,\Big\{\,\Gamma_{\beta}\,\Gamma^{\mu\nu\alpha\delta} \,-4\,\delta^{[\mu}_{\beta}\Gamma^{\nu\alpha\delta]}\,\Big\} \\
&= \quad \frac{(-1)^j\,(D-2\,j)}{4}\,\Big[\,\Gamma^{\mu\nu\alpha\delta},\,\Gamma^{(j)}\,\Big] \,-\, \Gamma^{[\mu\nu\alpha|}\,\Gamma^{(j)}\,\Gamma^{|\delta]} \,+\, \Gamma^{[\mu|}\,\Gamma^{(j)}\,\Gamma^{|\nu\alpha\delta]}
\end{aligned}
\tag{2.81}
$$

Final checks:

With the help of (2.80) and (2.81) equation (2.73) now reads

$$\Big\{\ j^{\text{th}} - \text{term},\ \text{c.p.}\Big\} \;=\; -\frac{1}{8}\,\big[\,(D-2\,j)^2 - D + 8 - 2\,(-1)^j\,(D-2\,j)\,\big]\,\Big[\,\Gamma^{\mu\nu\alpha\delta},\,\Gamma^{(j)}\,\Big] + \big[\,(-1)^j\,(D-2\,j)\,-\,1\,+\,2\,\big]\,\Big\{\Gamma^{[\mu\nu\alpha|}\,\Gamma^{(j)}\,\Gamma^{|\delta]}\,-\,\Gamma^{[\mu|}\,\Gamma^{(j)}\,\Gamma^{|\nu\alpha\delta]}\Big\} \tag{2.82}$$

This must now be evaluated for all the three values j might take on.

$\underline{\underline{j\,=\,1:}}$ $\quad\Rightarrow\quad \Gamma^{(j)}\,=\,\Gamma_{c_1}$ and (2.82) reads

$$\Big\{\ 1^{\text{st}} - \text{term},\ \text{c.p.}\Big\} \;=\; -\,12\,\big[\,\Gamma^{\mu\nu\alpha\delta},\,\Gamma_{c_1}\,\big]\,-\,8\,\Big\{\Gamma^{[\mu\nu\alpha}\,\Gamma_{c_1}\,\Gamma^{\delta]}\,-\,\Gamma^{[\mu}\,\Gamma_{c_1}\,\Gamma^{\nu\alpha\delta]}\Big\}$$

With

$$\big[\,\Gamma^{\mu\nu\alpha\delta},\,\Gamma_{c_1}\,\big] \;=\; -\,8\,\delta^{[\mu}_{c_1}\,\Gamma^{\nu\alpha\delta]}$$

$$\begin{aligned}
\Big\{\ 1^{\text{st}} - \text{term},\ \text{c.p.}\Big\} &= 96\,\delta^{[\mu}_{c_1}\,\Gamma^{\nu\alpha\delta]}\,-\,8\,\Big\{\Gamma^{[\mu\nu\alpha}\,\Gamma_{c_1}\,\Gamma^{\delta]}\,-\,\Gamma^{[\mu}\,\Gamma_{c_1}\,\Gamma^{\nu\alpha\delta]}\Big\}\\
&= 96\,\delta^{[\mu}_{c_1}\,\Gamma^{\nu\alpha\delta]}\,-\,8\,\Big\{\,\Gamma^{[\mu\nu\alpha}\,\Big(-\,\Gamma^{\delta]}\,\Gamma_{c_1}\,+\,2\,\delta^{\delta]}_{c_1}\Big)\,-\,\Big(-\,\Gamma_{c_1}\Gamma^{[\mu}\,+\,2\,\delta^{[\mu}_{c_1}\Big)\,\Gamma^{\nu\alpha\delta]}\,\Big\}\\
&= (\,96+16+16-64\,)\,\delta^{[\mu}_{c_1}\,\Gamma^{\nu\alpha\delta]}\;=\;64\,\delta^{[\mu}_{c_1}\,\Gamma^{\nu\alpha\delta]}
\end{aligned}$$

This 64 exactly matches with opposite sign the only closed term presented in (2.72).

$\underline{\underline{j\,=\,2:}}$ $\quad\Rightarrow\quad \Gamma^{(j)}\,=\,\Gamma_{c_1c_2}$ and (2.82) reads

$$\Big\{\ 2^{\text{nd}} - \text{term},\ \text{c.p.}\Big\} \;=\; -\,4\,\big[\,\Gamma^{\mu\nu\alpha\delta},\,\Gamma_{c_1c_2}\,\big]\,+\,8\,\Big\{\,\underbrace{\Gamma^{[\mu\nu\alpha|}\,\Gamma_{c_1c_2}\,\Gamma^{|\delta]}\,-\,\Gamma^{[\mu|}\,\Gamma_{c_1c_2}\,\Gamma^{|\nu\alpha\delta]}}_{(*)}\,\Big\}$$

With

$$\big[\,\Gamma^{\mu\nu\alpha\delta},\,\Gamma_{c_1c_2}\,\big] \;=\; 16\,\delta^{\mu}_{[c_1}\,\Gamma_{c_2]}{}^{\nu\alpha\delta]}$$

and

$$
\begin{aligned}
(*) \quad &= \; - g^{\beta[\alpha}\,\Gamma^{\delta\mu\nu]}\,\Gamma_{c_1c_2}\,\Gamma_\beta \;-\; \Gamma_\beta\,\Gamma_{c_1c_2}\,g^{\beta[\alpha}\,\Gamma^{\delta\mu\nu]} \\
&= \; - g^{\beta[\alpha}\,\Gamma^{\delta\mu\nu]}\,\Big\{\Gamma_{c_1c_2\beta} \;+\; 2\,\eta_{\beta[c_2}\Gamma_{c_1]}\Big\} \;-\; \Big\{\Gamma_{\beta c_1c_2} \;+\; 2\,\eta_{\beta[c_1}\Gamma_{c_2]}\Big\}\; g^{\beta[\alpha}\,\Gamma^{\delta\mu\nu]} \\
&= \; - g^{\beta[\alpha}\,\Big\{\Gamma^{\delta\mu\nu]}{}_{c_1c_2\beta} \;+\; 9\,\delta^{\nu}_{[c_1}\Gamma^{\delta\mu]}{}_{c_2\beta]} \;+\; 18\,\delta^{\nu}_{[c_1}\delta^{\mu}_{c_2}\Gamma^{\delta]}{}_{\beta]} \;+\; 6\,\delta^{\nu}_{[c_1}\delta^{\mu}_{c_2}\delta^{\delta]}_{\beta]}\Big\} \\
&\quad\; - g^{\beta[\alpha}\,\Big\{\Gamma_{\beta c_1c_2}{}^{\delta\mu\nu]} \;+\; 9\,\delta^{\delta}_{[c_2}\Gamma_{\beta c_1]}{}^{\mu\nu]} \;+\; 18\,\delta^{\delta}_{[c_2}\delta^{\mu}_{c_1}\Gamma_{\beta]}{}^{\nu]} + 6\,\delta^{\delta}_{[c_2}\delta^{\mu}_{c_1}\delta^{\nu]}_{\beta]}\Big\} \\
&\quad\; -2\,g^{\beta[\alpha}\,\Gamma^{\delta\mu\nu]}\,\eta_{\beta[c_2}\,\Gamma_{c_1]} \;-\; 2\,\eta_{\beta[c_1}\,\Gamma_{c_2]}\,g^{\beta[\alpha}\,\Gamma^{\delta\mu\nu]} \\
&= \; - 18\,g^{\beta[\alpha}\delta^{\nu}_{[c_1}\,\Gamma^{\delta\mu]}{}_{c_2\beta]} - 2\,g^{\beta[\alpha}\,\Gamma^{\delta\mu\nu]}\,\eta_{\beta[c_2}\,\Gamma_{c_1]} \;-\; 2\,\eta_{\beta[c_1}\,\Gamma_{c_2]}\,g^{\beta[\alpha}\,\Gamma^{\delta\mu\nu]} \\
&= \; - 18\,g^{\beta[\alpha}\delta^{\nu}_{[c_1}\,\Gamma^{\delta\mu]}{}_{c_2\beta]} - 2\,g^{\beta[\alpha}\,\eta_{\beta[c_2}\,\Big\{\Gamma^{\delta\mu\nu]}{}_{c_1]} \;+\; 3\,\delta^{\nu}_{c_1]}\Gamma^{\delta\mu]}\Big\} \\
&\qquad\qquad\qquad\qquad -2\,\eta_{\beta[c_1}\,g^{\beta[\alpha}\,\Big\{\Gamma_{c_2]}{}^{\delta\mu\nu]} \;+\; 3\,\delta^{\delta}_{c_2]}\Gamma^{\mu\nu]}\Big\} \\
&= \; - 18\,g^{\beta[\alpha}\delta^{\nu}_{[c_1}\,\Gamma^{\delta\mu]}{}_{c_2\beta]} - 4\,\delta^{[\alpha}_{[c_2}\,\Gamma^{\delta\mu\nu]}{}_{c_1]} \\
&= \; 12\,\delta^{[\nu}_{[c_1}\,\Gamma^{\delta\mu\alpha]}{}_{c_2]} - 4\,\delta^{[\alpha}_{[c_2}\,\Gamma^{\delta\mu\nu]}{}_{c_1]} \\
&= \; 8\,\delta^{[\mu}_{[c_1}\,\Gamma_{c_2]}{}^{\nu\alpha\delta]}
\end{aligned}
$$

we obtain

$$
\Big\{\;2^{\mathrm{nd}} - \mathrm{term},\;\; \mathrm{c.p.}\Big\} \;\;=\;\; (\,-64 \;+\; 64\,)\;\delta^{\mu}_{[c_1}\,\Gamma_{c_2]}{}^{\nu\alpha\delta]} \;\stackrel{!}{=}\; 0.
$$

$\underline{\underline{j\,=\,5:}}$ $\quad\Rightarrow\quad \Gamma^{(j)} \,=\, \Gamma^{c_1\ldots c_5}$ $\qquad$ This case is left as an exercise.

$\square$

Homework:

Exercise 2. *Complete the above proof by doing the case* $j\,=\,5$.

3 Eleven dimensional Supergravity à la Cremmer-Julia-Scherk

In a seminal paper Cremmer, Julia and Scherk derived the Lagrangian of the unique 11d supergravity nowadays mostly referred to as the low energy limit of M-theory [2]. It is an eleven dimensional theory of gravity involving a set of massless fields in eleven dimensions which carry a representation of supersymmetry. Since supersymmetry assigns to each bosonic degree of freedom a corresponding fermionic one and vice versa a consequence of the last property is that the number of physical degrees of freedom for fermionic and bosonic fields must match. Physical degrees of freedom for massless particles can be counted by choosing light cone gauge, i.e. by classifying the massless fields according to the little group $SO(9)$. A very nice physical discussion of how this counting works in detail can be found in [18]. Massless fields must transform as irreducible representations of the little group.
Being a theory of gravity, the metric $g_{\mu\nu}$ should appear in the set of fields. The same must be true for the gravitino ψ^α_μ, which is the superpartner of the metric. The corresponding degrees of freedom of metric and gravitino have to form irreducible representations of tensor products of $SO(9)$. The metric is a symmetric tensor with two indices and thus contained in

- $[\mathbf{9}_V \otimes \mathbf{9}_V]_{\text{symm}} = [\mathbf{1} \oplus \mathbf{36} \oplus \mathbf{44}]_{\text{symm}} = \mathbf{1} \oplus \mathbf{44}$.

Here the **1** corresponds to the trace and the **44** to the symmetric traceless tensor. The metric is identified with the **44**, while the **1**, usually called dilaton, does not occur as a dynamical field in eleven dimensional supergravity. Similarly the gravitino is contained in the tensor product of the fundamental vector representation and the spinor representation of $SO(9)$, which itself is not irreducible but decomposes into two irreducible components:

- $\mathbf{16}_S \otimes \mathbf{9}_V = \mathbf{16}_S \oplus \mathbf{128}$

Obviously, the gravtino belongs to the **128**.
Supersymmetry requires matching of bosonic and fermionic degrees of freedom on-shell. The big mismatch between **44** and **128** seems to require the introduction of another bosonic field. It turns out that the product of three vector representation contains a **84** dimensional representation in its totally antisymmetric part

- $[\mathbf{9}_V \otimes \mathbf{9}_V \otimes \mathbf{9}_V]_{\text{antisym}} = [\mathbf{36} \otimes \mathbf{9}_V]_{\text{antisym}} = [\mathbf{84} \oplus \mathbf{9} \oplus \mathbf{231}]_{\text{antisym}} = \mathbf{84}$

Using this three form potential, the boson/fermion count now reads

$$\mathbf{44} \oplus \mathbf{84} = \mathbf{128}$$

From the point of view of counting the degrees of freedom there is a chance to find a supersymmetric theory containing the metric, a gravitino and a three form antisymmetric tensor field $C_{\mu\nu\rho}$ subject to suitable constraints in order to get rid of superfluous degrees of freedom as discussed before. The

construction of the theory from scratch will be postponed until section 4. At this point, we make do with giving the final result for the Lagrangian whose details depends on the conventions chosen. The vast majority of research papers uses conventions which differ from the ones used in the original paper [2]. Using the following redefinitions

- $\kappa = 1$
- $F_{\mu\nu\rho\sigma} = \frac{1}{2} G_{\mu\nu\rho\sigma}$
- $\Gamma^a \mapsto i\Gamma^a$

one obtains from the original Lagrangian (see appendix A) the Lagrangian in our conventions*6

$$\begin{aligned}
\mathcal{L} \quad = \quad & \frac{1}{4}\, eR + \frac{1}{2}\, e\bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu \left(\frac{\omega + \hat{\omega}}{2}\right) \psi_\rho - \frac{1}{4 \cdot 48}\, eG_{\mu\nu\rho\sigma} G^{\mu\nu\rho\sigma} \\
& - \frac{1}{4 \cdot 48}\, e \left(\bar{\psi}_\mu \Gamma^{\mu\nu\alpha\beta\gamma\delta} \psi_\nu + 12\, \bar{\psi}^\alpha \Gamma^{\gamma\delta} \psi^\beta\right) \left(\frac{G_{\alpha\beta\gamma\delta} + \hat{G}_{\alpha\beta\gamma\delta}}{2}\right) \\
& + \frac{1}{4 \cdot 144^2}\, \epsilon^{\alpha_1 \ldots \alpha_4 \beta_1 \ldots \beta_4 \mu\nu\rho} G_{\alpha_1 \ldots \alpha_4} G_{\beta_1 \ldots \beta_4} C_{\mu\nu\rho}\ .
\end{aligned} \tag{3.1}$$

the action of which is supposed to be invariant under the following supersymmetric transformation rules:

$$\begin{aligned}
\delta_Q e_\mu{}^a \quad & = \quad \bar{\epsilon}\, \Gamma^a \psi_\mu \\
\delta_Q C_{\mu\nu\rho} \quad & = \quad 3\, \bar{\epsilon}\, \Gamma_{[\mu\nu} \psi_{\rho]} \\
\delta_Q \psi_\mu \quad & = \quad D_\mu(\hat{\omega})\epsilon - \frac{1}{2 \cdot 144} \left(\Gamma^{\alpha\beta\gamma\delta}{}_\mu - 8\, \Gamma^{\beta\gamma\delta} \delta^\alpha_\mu\right) \epsilon\, \hat{G}_{\alpha\beta\gamma\delta} \;=\; \hat{D}_\mu \epsilon
\end{aligned} \tag{3.2}$$

The signature of the metric is $\eta_{ab} = (-1, 1, \ldots, 1)$ and the Γ-matrices are in a real representation of the Clifford algebra

$$\{\, \Gamma^a,\, \Gamma^b \,\} \quad = \quad 2\, \eta^{ab}\, \mathbb{1}_{32}\ .$$

Apart from the elementary fields (vielbein $e^a{}_\mu$, gravitino ψ_μ and antisymmetric tensor field $A_{\mu\nu\rho}$) the abbreviations in the above Lagrangian have the following meaning:

*6 $\varepsilon^{0123456789\,10} = -1$

$$
\begin{aligned}
D_\nu(\omega)\psi_\mu &= \partial_\nu \psi_\mu - \frac{1}{4}\,\omega_{\nu ab}\,\Gamma^{ab}\,\psi_\mu \\
G_{\mu\nu\rho\sigma} &= 4\,\partial_{[\mu}\,C_{\nu\rho\sigma]} \\
\hat{G}_{\mu\nu\rho\sigma} &= G_{\mu\nu\rho\sigma} \;+\; 6\,\bar{\psi}_{[\mu}\Gamma_{\nu\rho}\psi_{\sigma]} \\
K_{\mu ab} &= \frac{1}{4}\,\Big[\,\bar{\psi}_\alpha \Gamma_{\mu ab}{}^{\alpha\beta}\psi_\beta \;-\; 2\,\big(\,\bar{\psi}_\mu\Gamma_b\psi_a \;-\; \bar{\psi}_\mu\Gamma_a\psi_b \;+\; \bar{\psi}_b\Gamma_\mu\psi_a\,\big)\Big] \qquad (\,\text{contorsion}\,) \\
\omega_{\mu ab} &= \underbrace{\omega^{(0)}_{\mu ab}}_{\text{Christ}} \;+\; K_{\mu ab} \\
\hat{\omega}_{\mu ab} &= \omega_{\mu ab} \;-\; \frac{1}{4}\,\bar{\psi}_\alpha\,\Gamma_{\mu ab}{}^{\alpha\beta}\,\psi_\beta
\end{aligned}
$$

3.1 Equation of Motion

3.1.1 ψ_μ

$$\mathcal{L}_\psi \;=\; \frac{1}{2}\,\bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu \left(\frac{\omega \,+\, \hat{\omega}}{2}\right) \psi_\rho \;-\; \frac{1}{4\cdot 48}\,\left(\bar{\psi}_\mu \Gamma^{\mu\nu\alpha\beta\gamma\delta}\psi_\nu + 12\,\bar{\psi}^\alpha \Gamma^{\gamma\delta}\psi^\beta\right)\,\left(\hat{G}_{\alpha\beta\gamma\delta} \,-\, 3\,\bar{\psi}_{[\alpha}\Gamma_{\beta\gamma}\psi_{\delta]}\right)$$

$$0 \;=\; \frac{\partial \mathcal{L}}{\partial \bar{\psi}^n_\xi} \;-\; \frac{\partial}{\partial x^\omega}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\omega \bar{\psi}^n_\xi\right)}\right) \;\stackrel{!}{=}\; \frac{\partial \mathcal{L}}{\partial \bar{\psi}^n_\xi}$$

$$\begin{aligned}
0 \;=\;& \frac{1}{2}\,\Gamma^{\xi\nu\rho}\left[D_\nu(\hat{\omega}) \,-\, \frac{1}{32}\,\bar{\psi}_\alpha \Gamma_{\nu ab}{}^{\alpha\beta}\,\psi_\beta\,\Gamma^{ab}\right]\psi_\rho \;+\; \frac{1}{64}\,\Gamma_{\nu ab}{}^{\xi\beta}\,\psi_\beta\,\bar{\psi}_\mu\,\Gamma^{\mu\nu\rho}\,\Gamma^{ab}\,\psi_\rho \\
&-\; \frac{1}{4\cdot 48}\,\left(\Gamma^{\xi\nu}{}_{\alpha\beta\gamma\delta}\,\psi_\nu \;+\; 12\,\delta^\xi_\alpha\,\Gamma_{\gamma\delta}\psi_\beta\right)\,\left(\hat{G}^{\alpha\beta\gamma\delta} \,-\, 3\,\bar{\psi}^{[\alpha}\Gamma^{\beta\gamma}\psi^{\delta]}\right) \\
&-\; \frac{3}{4\cdot 48}\,\delta^\xi_{[\alpha}\Gamma_{\beta\gamma}\psi_{\delta]}\,\left(\bar{\psi}_\mu\,\Gamma^{\mu\nu\alpha\beta\gamma\delta}\,\psi_\nu \;+\; 12\,\bar{\psi}^\alpha\,\Gamma^{\gamma\delta}\psi^\beta\right)
\end{aligned}$$

$$\begin{aligned}
=\;& \frac{1}{2}\left[\Gamma^{\xi\nu\rho}\,D_\nu(\hat{\omega}) \,-\, \frac{1}{96}\,\left(\Gamma^{\xi\rho}{}_{\alpha\beta\gamma\delta} \;+\; 12\,\delta^\xi_\alpha\,\Gamma_{\gamma\delta}\delta^\rho_\beta\right)\,\hat{G}^{\alpha\beta\gamma\delta}\right]\psi_\rho \\
&-\; \frac{1}{64}\,\Gamma^{\xi\nu\rho}\,\Gamma^{ab}\,\psi_\rho\,\bar{\psi}_\alpha\Gamma_{\nu ab}{}^{\alpha\beta}\,\psi_\beta \;+\; \frac{1}{64}\,\Gamma_{\nu ab}{}^{\xi\beta}\,\psi_\beta\,\bar{\psi}_\mu\,\Gamma^{\mu\nu\rho}\,\Gamma^{ab}\,\psi_\rho \\
&+\; \frac{1}{64}\,\left(\Gamma^{\xi\nu}{}_{\alpha\beta\gamma\delta}\,\psi_\nu\,\bar{\psi}^{[\alpha}\Gamma^{\beta\gamma}\psi^{\delta]} \;-\; \delta^\xi_{[\alpha}\Gamma_{\beta\gamma}\psi_{\delta]}\,\bar{\psi}_\mu\,\Gamma^{\mu\nu\alpha\beta\gamma\delta}\,\psi_\nu\right)
\end{aligned} \tag{3.3}$$

Proposition 7.

$$\Gamma^{\mu\nu\rho}\,\left(\Gamma^{\alpha\beta\gamma\delta}{}_\nu - 8\,\Gamma^{\beta\gamma\delta}\,\delta^\alpha_\nu\right)\,\psi_\rho\,\hat{G}_{\alpha\beta\gamma\delta} \;=\; 3\,\left(\Gamma^{\mu\rho}{}_{\alpha\beta\gamma\delta} \;+\; 12\,\delta^\mu_\alpha\,\Gamma_{\gamma\delta}\delta^\rho_\beta\right)\,\psi_\rho\,\hat{G}^{\alpha\beta\gamma\delta}$$

Proof. We quote two useful identities here. The first is simply a variant of (2.53).

$$\Gamma^a\Gamma^{a_1\ldots a_j} \;=\; \Gamma^{aa_1\ldots a_j} \;+\; j\,\eta^{a[a_1}\Gamma^{a_2\ldots a_j]} \tag{3.4}$$

$$\Gamma^d\Gamma_{a_1\ldots a_j}\Gamma_d \;=\; (-1)^j\,(11-2j)\,\Gamma_{a_1\ldots a_j} \tag{3.5}$$

Using eq. (2.53) and eq. (3.4) repeatedly one obtains

$$
\begin{aligned}
\Gamma^{\mu\nu\rho}\,\Gamma_{\alpha\beta\gamma\delta\nu} &= -\Gamma^{\nu\mu\rho}\,\Gamma_{\alpha\beta\gamma\delta\nu} \\
&= -\left(\,\Gamma^{\nu}\,\Gamma^{\mu\rho} \,-\, 2\,\eta^{\nu[\mu}\,\Gamma^{\rho]}\,\right)\left(\,\Gamma_{\alpha\beta\gamma\delta}\Gamma_{\nu} \,-\, 4\,\Gamma_{[\alpha\beta\gamma}\eta_{\delta]\nu}\,\right) \\
&= -\,\Gamma^{\nu}\,\Gamma^{\mu\rho}\,\Gamma_{\alpha\beta\gamma\delta}\Gamma_{\nu} \;+\; 4\,\Gamma^{\nu}\,\Gamma^{\mu\rho}\,\Gamma_{[\alpha\beta\gamma}\eta_{\delta]\nu} \\
&\quad +2\,\eta^{\nu[\mu}\,\Gamma^{\rho]}\,\Gamma_{\alpha\beta\gamma\delta}\Gamma_{\nu} \;-\; 8\,\eta^{\nu[\mu}\,\Gamma^{\rho]}\,\Gamma_{[\alpha\beta\gamma}\eta_{\delta]\nu} \\
&= -\,\Gamma^{\nu}\,\left\{\,\Gamma^{\mu\rho}{}_{\alpha\beta\gamma\delta} \,+\, 8\,\delta^{[\rho}_{[\alpha}\,\Gamma^{\mu]}{}_{\beta\gamma\delta]} \,+\, 12\,\delta^{[\rho}_{[\alpha}\delta^{\mu]}_{\beta}\Gamma_{\gamma\delta]}\,\right\}\Gamma_{\nu} \\
&\quad +4\,\left\{\,\Gamma^{\nu\mu\rho} \,+\, 2\,\eta^{\nu[\mu}\,\Gamma^{\rho]}\,\right\}\Gamma_{[\alpha\beta\gamma}\eta_{\delta]\nu} \;+\; 2\,\eta^{\nu[\mu}\,\Gamma^{\rho]}\,\Gamma_{\alpha\beta\gamma\delta}\Gamma_{\nu} \\
&\quad -8\,\eta^{\nu[\mu}\,\Gamma^{\rho]}\,\Gamma_{[\alpha\beta\gamma}\eta_{\delta]\nu} \\
&= +\,\Gamma^{\mu\rho}{}_{\alpha\beta\gamma\delta} \,-\, 24\,\delta^{[\rho}_{[\alpha}\,\Gamma^{\mu]}{}_{\beta\gamma\delta]} \,-\, 84\,\delta^{[\rho}_{[\alpha}\delta^{\mu]}_{\beta}\Gamma_{\gamma\delta]} \\
&\quad +4\,\Gamma^{\nu\mu\rho}\,\Gamma_{[\alpha\beta\gamma}\eta_{\delta]\nu} \;+\; 2\,\eta^{\nu[\mu}\,\Gamma^{\rho]}\,\Gamma_{\alpha\beta\gamma\delta}\Gamma_{\nu}
\end{aligned}
$$

Now contracting with $\hat{G}^{\alpha\beta\gamma\delta}$ one obtains

$$
\begin{aligned}
\Gamma^{\mu\nu\rho}\,\Gamma_{\alpha\beta\gamma\delta\nu}\,\hat{G}^{\alpha\beta\gamma\delta} &= \Big\{\,\Gamma^{\mu\rho}{}_{\alpha\beta\gamma\delta} \,-\, 24\,\delta^{[\rho}_{\alpha}\,\Gamma^{\mu]}{}_{\beta\gamma\delta} \,-\, 84\,\delta^{[\rho}_{\alpha}\delta^{\mu]}_{\beta}\Gamma_{\gamma\delta} \\
&\quad -4\,\Gamma^{\nu\mu\rho}\,\Gamma_{\beta\gamma\delta}\eta_{\alpha\nu} \;+\; 2\,\eta^{\nu[\mu}\,\Gamma^{\rho]}\,\Gamma_{\alpha\beta\gamma\delta}\Gamma_{\nu}\,\Big\}\,\hat{G}^{\alpha\beta\gamma\delta}
\end{aligned}
$$

The explicit antisymmetrisation in $(\alpha,\beta,\gamma,\delta)$ of the second, third and fourth term could be neglected, since we contract via an complete antisymmetric tensor $\hat{G}^{\alpha\beta\gamma\delta}$. We consider now

$$
\begin{aligned}
\Gamma^{\mu\nu\rho}\left(\Gamma_{\alpha\beta\gamma\delta\nu} - 8\,\Gamma_{\beta\gamma\delta}\,\eta_{\nu\alpha}\right)\,\hat{G}^{\alpha\beta\gamma\delta} &= \Big\{\,\Gamma^{\mu\rho}{}_{\alpha\beta\gamma\delta} \,-\, 24\,\delta^{[\rho}_{\alpha}\,\Gamma^{\mu]}{}_{\beta\gamma\delta} \,-\, 84\,\delta^{[\rho}_{\alpha}\delta^{\mu]}_{\beta}\Gamma_{\gamma\delta} \\
&\quad -4\,\Gamma^{\mu\nu\rho}\,\Gamma_{\beta\gamma\delta}\eta_{\alpha\nu} \;+\; 2\,\eta^{\nu[\mu}\,\Gamma^{\rho]}\,\Gamma_{\alpha\beta\gamma\delta}\Gamma_{\nu}\,\Big\}\,\hat{G}^{\alpha\beta\gamma\delta}
\end{aligned}
\tag{3.6}
$$

Expanding the fourth term of eq. (3.6) via eq. (2.53) one obtains:

$$
\Gamma^{\mu\nu\rho}\Gamma_{\beta\gamma\delta}\,\eta_{\nu\alpha} = \Gamma^{\mu}{}_{\alpha}{}^{\rho}{}_{\beta\gamma\delta} \;+\; 9\,\eta_{\nu\alpha}\,\delta^{[\rho}_{[\beta}\Gamma^{\mu\nu]}{}_{\gamma\delta]} \;+\; 18\,\eta_{\nu\alpha}\,\delta^{[\rho}_{[\beta}\delta^{\nu}_{\gamma}\Gamma^{\mu]}{}_{\delta]} \;+\; 6\eta_{\nu\alpha}\,\delta^{[\mu}_{[\beta}\delta^{\nu}_{\gamma}\delta^{\rho]}_{\delta]}
$$

The last term of eq. (3.6) can be expanded as

$$
\begin{aligned}
2\,\eta^{\nu[\mu}\,\Gamma^{\rho]}\,\Gamma_{\alpha\beta\gamma\delta}\Gamma_\nu\,\hat{G}^{\alpha\beta\gamma\delta} &= 2\,\eta^{\nu[\mu}\,\Gamma^{\rho]}\,\Big\{\,\Gamma_{\alpha\beta\gamma\delta\nu}\,+\,4\,\Gamma_{[\alpha\beta\gamma}\eta_{\delta]\nu}\,\Big\}\,\hat{G}^{\alpha\beta\gamma\delta}\\
&= 2\,\eta^{\nu[\mu}\,\delta^{\rho]}_\kappa\,\Gamma^\kappa\,\Big\{\,\Gamma_{\alpha\beta\gamma\delta\nu}\,+\,4\,\Gamma_{\alpha\beta\gamma}\eta_{\delta\nu}\,\Big\}\,\hat{G}^{\alpha\beta\gamma\delta}\\
&= 2\,\eta^{\nu[\mu}\,\delta^{\rho]}_\kappa\,\Big\{\,\Gamma^\kappa{}_{\alpha\beta\gamma\delta\nu}\,+\,5\delta^\kappa_{[\alpha}\,\Gamma_{\beta\gamma\delta\nu]}\,+\,4\,\Big(\Gamma^\kappa{}_{\alpha\beta\gamma}\,+\,3\delta^\kappa_{[\alpha}\Gamma_{\beta\gamma]}\Big)\,\eta_{\delta\nu}\,\Big\}\,\hat{G}^{\alpha\beta\gamma\delta}\\
&= 2\,\eta^{\nu[\mu}\,\delta^{\rho]}_\kappa\,\Big\{\,\Gamma^\kappa{}_{\alpha\beta\gamma\delta\nu}\,+\,4\,(\,\Gamma^\kappa{}_{\alpha\beta\gamma}\,+\,3\delta^\kappa_\alpha\Gamma_{\beta\gamma}\,)\,\eta_{\delta\nu}\,+\,\big(\,\delta^\kappa_\alpha\,\Gamma_{\beta\gamma\delta\nu}\,+\,\underbrace{\delta^\kappa_\nu\,\Gamma_{\alpha\beta\gamma\delta}}_{0}\\
&\qquad\qquad +\,\delta^\kappa_\delta\,\Gamma_{\nu\alpha\beta\gamma}\,+\,\delta^\kappa_\gamma\,\Gamma_{\delta\nu\alpha\beta}\,+\,\delta^\kappa_\beta\,\Gamma_{\gamma\delta\nu\alpha}\,\big)\,\Big\}\,\hat{G}^{\alpha\beta\gamma\delta}\\
&= \Big\{\,-\,2\,\Gamma^{\mu\rho}{}_{\alpha\beta\gamma\delta}\,+\,8\,\delta^{[\mu}_\delta\,\delta^{\rho]}_\kappa\,(\,\Gamma^\kappa{}_{\alpha\beta\gamma}\,+\,3\delta^\kappa_\alpha\Gamma_{\beta\gamma}\,)\,+\,8\,\eta^{\nu[\mu}\,\delta^{\rho]}_\kappa\,\delta^\kappa_{[\alpha}\,\Gamma_{\beta\gamma\delta]\nu}\,\Big\}\,\hat{G}^{\alpha\beta\gamma\delta}\\
&= \Big\{\,-\,2\,\Gamma^{\mu\rho}{}_{\alpha\beta\gamma\delta}\,+\,8\,\delta^{[\mu}_\delta\,\Gamma^{\rho]}{}_{\alpha\beta\gamma}\,+\,24\,\delta^{[\mu}_\delta\,\delta^{\rho]}_\alpha\,\Gamma_{\beta\gamma}\,-\,8\,\delta^{[\mu}_\delta\,\Gamma^{\rho]}{}_{\alpha\beta\gamma}\,\Big\}\,\hat{G}^{\alpha\beta\gamma\delta}\\
&= \Big\{\,-\,2\,\Gamma^{\mu\rho}{}_{\alpha\beta\gamma\delta}\,+\,24\,\delta^{[\mu}_\delta\,\delta^{\rho]}_\alpha\,\Gamma_{\beta\gamma}\,\Big\}\,\hat{G}^{\alpha\beta\gamma\delta}
\end{aligned}
$$

Putting things together eq. (3.6) reads

$$
\begin{aligned}
\Gamma^{\mu\nu\rho}\,(\Gamma_{\alpha\beta\gamma\delta\nu}-8\,\Gamma_{\beta\gamma\delta}\,\eta_{\nu\alpha})\,\hat{G}^{\alpha\beta\gamma\delta} &= \Big\{\,\Gamma^{\mu\rho}{}_{\alpha\beta\gamma\delta}\,-\,24\,\delta^{[\rho}_\alpha\,\Gamma^{\mu]}{}_{\beta\gamma\delta}\,-\,84\,\delta^{[\rho}_\alpha\delta^{\mu]}_\beta\Gamma_{\gamma\delta}\\
&\quad -4\,\Big(\,\Gamma^\mu{}_\alpha{}^\rho{}_{\beta\gamma\delta}\,+\,9\,\eta_{\nu\alpha}\,\delta^{[\rho}_{[\beta}\Gamma^{\mu\nu]}{}_{\gamma\delta]}\,+\,18\,\eta_{\nu\alpha}\,\delta^{[\rho}_{[\beta}\delta^\nu_\gamma\Gamma^{\mu]}{}_{\delta]}\,+\,\underbrace{6\eta_{\nu\alpha}\,\delta^{[\mu}_{[\beta}\delta^\nu_\gamma\delta^{\rho]}_{\delta]}}_{0}\Big)\\
&\quad +\,\Big(-\,2\,\Gamma^{\mu\rho}{}_{\alpha\beta\gamma\delta}\,+\,24\,\delta^{[\mu}_\delta\,\delta^{\rho]}_\alpha\,\Gamma_{\beta\gamma}\Big)\,\Big\}\,\hat{G}^{\alpha\beta\gamma\delta}\\
&= \Big\{\,3\,\Gamma^{\mu\rho}{}_{\alpha\beta\gamma\delta}\,-\,84\,\delta^{[\rho}_\alpha\delta^{\mu]}_\beta\Gamma_{\gamma\delta}\,-\,72\,\eta_{\nu\alpha}\,\delta^{[\rho}_\beta\delta^\nu_\gamma\Gamma^{\mu]}{}_\delta\,+\,24\,\delta^{[\mu}_\delta\,\delta^{\rho]}_\alpha\,\Gamma_{\beta\gamma}\\
&\quad -\,24\,\delta^{[\rho}_\alpha\,\Gamma^{\mu]}{}_{\beta\gamma\delta}\,-\,36\,\eta_{\nu\alpha}\,\delta^{[\rho}_\beta\Gamma^{\mu\nu]}{}_{\gamma\delta}\,\Big\}\,\hat{G}^{\alpha\beta\gamma\delta}
\end{aligned}
\tag{3.7}
$$

Now a case by case study of the remaining terms follows

$$
\begin{aligned}
-\,36\,\eta_{\nu\alpha}\,\delta^{[\rho}_\beta\Gamma^{\mu\nu]}{}_{\gamma\delta}\,\hat{G}^{\alpha\beta\gamma\delta} &= -\,12\,\eta_{\nu\alpha}\,\Big[\,\delta^\rho_\beta\Gamma^{\mu\nu}{}_{\gamma\delta}\,+\,\delta^\mu_\beta\Gamma^{\nu\rho}{}_{\gamma\delta}\,+\,\delta^\nu_\beta\Gamma^{\rho\mu}{}_{\gamma\delta}\,\Big]\,\hat{G}^{\alpha\beta\gamma\delta}\\
&= -\,12\,\Big[\,\delta^\rho_\beta\Gamma^\mu{}_{\alpha\gamma\delta}\,-\,\delta^\mu_\beta\Gamma^\rho{}_{\alpha\gamma\delta}\,+\,\underbrace{\eta_{\alpha\beta}\Gamma^{\rho\mu}{}_{\gamma\delta}}_{0}\,\Big]\,\hat{G}^{\alpha\beta\gamma\delta}\\
&= 12\,\Gamma^\mu{}_{\alpha\gamma\delta}\hat{G}^{\rho\alpha\gamma\delta}\,-\,12\,\Gamma^\rho{}_{\alpha\gamma\delta}\hat{G}^{\mu\alpha\gamma\delta}\\
&= 24\,\Gamma^{[\mu}{}_{\alpha\gamma\delta}\hat{G}^{\rho]\alpha\gamma\delta}\,=\,24\,\Gamma^{[\mu}{}_{\alpha\gamma\delta}\delta^{\rho]}_\beta\,\hat{G}^{\beta\alpha\gamma\delta}\,=\,24\,\delta^{[\rho}_\alpha\Gamma^{\mu]}{}_{\beta\gamma\delta}\hat{G}^{\alpha\beta\gamma\delta}
\end{aligned}
$$

$$
\begin{aligned}
-72\,\eta_{\nu\alpha}\,\delta^{[\rho}_{\beta}\delta^{\nu}_{\gamma}\Gamma^{\mu]}{}_{\delta}\,\hat{G}^{\alpha\beta\gamma\delta} &= -72\,\eta_{\nu\alpha}\,\frac{1}{3}\,\Big[\;\delta^{\rho}_{\beta}\delta^{\nu}_{\gamma}\Gamma^{\mu}{}_{\delta} \;+\; \delta^{\nu}_{\beta}\delta^{\mu}_{\gamma}\Gamma^{\rho}{}_{\delta} \;+\; \delta^{\mu}_{\beta}\delta^{\rho}_{\gamma}\Gamma^{\nu}{}_{\delta}\;\Big]\,\hat{G}^{\alpha\beta\gamma\delta} \\
&= -72\,\eta_{\nu\alpha}\,\frac{1}{3}\,\Big[\;\underbrace{\delta^{\rho}_{\beta}\delta^{\nu}_{\gamma}\Gamma^{\mu}{}_{\delta}}_{0} \;+\; \underbrace{\delta^{\nu}_{\beta}\delta^{\mu}_{\gamma}\Gamma^{\rho}{}_{\delta}}_{0} \;+\; \delta^{\mu}_{\beta}\delta^{\rho}_{\gamma}\Gamma^{\nu}{}_{\delta}\;\Big]\,\hat{G}^{\alpha\beta\gamma\delta} \\
&= -72\,\frac{1}{3}\,\delta^{\rho}_{\gamma}\delta^{\mu}_{\beta}\Gamma_{\alpha\delta}\,\hat{G}^{\alpha\beta\gamma\delta} \\
&= 24\,\delta^{\rho}_{\alpha}\delta^{\mu}_{\beta}\,\Gamma_{\gamma\delta}\,\hat{G}^{\alpha\beta\gamma\delta}
\end{aligned}
$$

Plugging this expressions in eq. (3.7) one obtains

$$
\Gamma^{\mu\nu\rho}\left(\Gamma_{\alpha\beta\gamma\delta\nu} - 8\,\Gamma_{\beta\gamma\delta}\,\eta_{\nu\alpha}\right)\hat{G}^{\alpha\beta\gamma\delta} \;=\; 3\cdot\Big(\;\Gamma^{\mu\rho}{}_{\alpha\beta\gamma\delta} + 12\,\delta^{\mu}_{\alpha}\,\Gamma_{\gamma\delta}\,\delta^{\rho}_{\beta}\;\Big)\,\hat{G}^{\alpha\beta\gamma\delta} \tag{3.8}
$$

A contraction of this equation with ψ_ρ gives the desired result.

□

Proposition 8. *The last four terms in eq. (3.3) vanish identically.*

Proof. The basic steps of the proof are outlined in [29] and boil down to checking that the Fierz identity (2.72) holds.

□

If one inserts the results of Proposition 1 and 2 in eq. (3.3) and defines

$$
\hat{D}_\nu\psi_\rho \;=\; D_\nu(\hat{\omega})\psi_\rho - \frac{1}{2\cdot 144}\left(\Gamma^{\alpha\beta\gamma\delta}{}_{\nu} - 8\,\Gamma^{\beta\gamma\delta}\delta^{\alpha}_{\nu}\right)\,\psi_\rho\,\hat{G}_{\alpha\beta\gamma\delta}
$$

the equation of motion finally reads

$$
\Gamma^{\mu\nu\rho}\hat{D}_\nu\psi_\rho \;=\; 0 \tag{3.9}
$$

Remark 2. In practical applications it is very cumbersome to work with the fermionic fields. So one mostly restricts considerations to the set of bosonic fields and sets the gravitino explicitly to zero. To make the vanishing of the gravitino consistent with the presence of supersymmetry one has to impose the constraint that the supervariations acting on such a bosonic background cannot restore a non vanishing gravitino, i.e. the supervariation of the gravitino evaluated in the bosonic background

must vanish identically:

$$\delta_Q \psi_\mu \quad = \quad D_\mu(\omega)\epsilon - \frac{1}{2 \cdot 144} \left(\Gamma^{\alpha\beta\gamma\delta}{}_\mu - 8\, \Gamma^{\beta\gamma\delta} \delta^\alpha_\mu \right) \epsilon \, G_{\alpha\beta\gamma\delta} \ \stackrel{!}{=} \ 0 \ . \tag{3.10}$$

This is known as the Killing spinor equation of eleven dimensional supergravity. Following the common practice we just work out the bosonic equations of motion of the metric $g_{\mu\nu}$ and three form potential $C_{\mu\nu\rho}$, now.

3.1.2 $g_{\mu\nu}$

$$S \;=\; \int d^D x\, \underbrace{\sqrt{g}\, R}_{(I)} \;-\; \frac{1}{48} \int d^D x\, \underbrace{\sqrt{g}\, G_{\mu\nu\rho\sigma}\, G^{\mu\nu\rho\sigma}}_{(II)} \tag{3.11}$$

We will now derive the Euler-Lagrange equations following from the Einstein-Hilbert action (I), coupled to a four form field strength (II). First we study the variation of the Einstein-Hilbert action (I). Since the Ricci scalar is $R = g^{\mu\nu} R_{\mu\nu}$, it is simpler to consider variations of $g^{\mu\nu}$, i.e. the inverse metric, instead of $g_{\mu\nu}$.

$$\delta S_{(I)} \;=\; \int d^D x \left[\, \delta\sqrt{g}\, g^{\mu\nu}\, R_{\mu\nu} \;+\; \sqrt{g}\, \delta g^{\mu\nu}\, R_{\mu\nu} \;+\; \sqrt{g}\, g^{\mu\nu}\, \delta R_{\mu\nu} \,\right] \tag{3.12}$$

We must evaluate the variation of the determinant in the the first term. We start from the identity

$$g \;\equiv\; -\det g \;=\; -\, \varepsilon^{i_1 \ldots i_n}\, g_{1 i_1} \cdot \ldots \cdot g_{n i_n}$$

and compute

$$\frac{\partial g}{\partial x^l} \;=\; -\sum_{j=1}^{n} \varepsilon^{i_1 \ldots i_n}\, g_{1 i_1} \cdot \ldots \cdot \frac{\partial g_{j i_j}}{\partial x^l} \cdot \ldots \cdot g_{n i_n}\ .$$

The derivative can be rewritten via

$$\frac{\partial g_{j i_j}}{\partial x^l} \;=\; \delta^k_{i_j}\, \frac{\partial g_{jk}}{\partial x^l} \;=\; g_{i_j m} g^{mk}\, \frac{\partial g_{jk}}{\partial x^l}\ .$$

Inserting this back into the previous equation we obtain

$$\begin{aligned} \frac{\partial g}{\partial x^l} \;&=\; -\sum_{j=1}^{n} \varepsilon^{i_1 \ldots i_n}\, g_{1 i_1} \cdot \ldots \cdot \Bigg(\underbrace{g_{i_j m} g^{mk}\, \frac{\partial g_{jk}}{\partial x^l}}_{m \stackrel{!}{=} j} \Bigg) \cdot \ldots \cdot g_{n i_n} \\ &=\; g^{jk}\, \frac{\partial g_{jk}}{\partial x^l} \cdot g \end{aligned} \tag{3.13}$$

Algebraically δ behaves like a derivation. Therefore we use eq. (3.13) and just replace the partial derivative by δ to obtain

$$\delta\sqrt{g} \;=\; \frac{1}{2}\,\frac{1}{\sqrt{g}}\, \delta g \;=\; \frac{1}{2}\,\frac{1}{\sqrt{g}}\, g\, g^{jk}\, \delta g_{jk} \;=\; \frac{1}{2}\, \sqrt{g}\, g^{jk}\, \delta g_{jk} \;=\; -\,\frac{1}{2}\, \sqrt{g}\, g_{jk}\, \delta g^{jk} \tag{3.14}$$

and eq. (3.12) simplifies to

$$\delta S_{(I)} \;=\; \int d^D x \, \sqrt{g} \left[\, - \, \frac{1}{2} \, g_{\mu\nu} \, R \; + \; R_{\mu\nu} \, \right] \delta g^{\mu\nu} \; + \; \int d^D x \, \sqrt{g} \, g^{\mu\nu} \, \delta R_{\mu\nu} \tag{3.15}$$

The first two terms in this expression already form the Einstein tensor. Therefore we must prove only that the third term does not contribute. To this purpose we have to evaluate the variation of the Ricci tensor, who is given by

$$R_{\mu\nu} \;=\; R^{\rho}{}_{\mu\rho\nu} \;=\; \frac{\partial \Gamma^{\lambda}{}_{(\mu\nu)}}{\partial x^{\lambda}} \;-\; \frac{\partial \Gamma^{\lambda}{}_{(\mu\lambda)}}{\partial x^{\nu}} \;+\; \Gamma^{\lambda}{}_{(\lambda\rho)} \, \Gamma^{\rho}{}_{(\nu\mu)} \;-\; \Gamma^{\lambda}{}_{(\nu\rho)} \, \Gamma^{\rho}{}_{(\lambda\mu)} \; .$$

There is an easy way to obtain the variation with respect to the metric. For this we compute the variation of $R_{\mu\nu}$ in terms of the variations $\delta\Gamma^{\mu}{}_{\nu\lambda}$ induced by the variations of the metric. So $\delta R_{\mu\nu}$ becomes

$$\begin{aligned} \delta R_{\mu\nu} \;=\; & \frac{\partial \delta\Gamma^{\lambda}{}_{(\mu\nu)}}{\partial x^{\lambda}} \;-\; \frac{\partial \delta\Gamma^{\lambda}{}_{(\mu\lambda)}}{\partial x^{\nu}} \;+\; \delta\Gamma^{\lambda}{}_{(\lambda\rho)} \, \Gamma^{\rho}{}_{(\nu\mu)} \;+\; \Gamma^{\lambda}{}_{(\lambda\rho)} \, \delta\Gamma^{\rho}{}_{(\nu\mu)} \\ & \;-\; \delta\Gamma^{\lambda}{}_{(\nu\rho)} \, \Gamma^{\rho}{}_{(\lambda\mu)} \;-\; \Gamma^{\lambda}{}_{(\nu\rho)} \, \delta\Gamma^{\rho}{}_{(\lambda\mu)} \end{aligned}$$

Now the crucial observation is that $\delta\Gamma^{\mu}{}_{(\nu\lambda)}$ is a tensor. The Christoffel symbols themselves are not tensors, because of the inhomogeneous (second derivative) term appearing in the transformation rule under coordinate transformations. But this term is independent of the metric. Thus the metric variation of the Christoffel symbols indeed transforms as a tensor, and it turns out that $\delta R_{\mu\nu}$ can be written rather compactly in terms of covariant derivatives of $\delta\Gamma^{\mu}{}_{(\nu\lambda)}$, namely as

$$\delta R_{\mu\nu} \;=\; \nabla_{\lambda} \delta\Gamma^{\lambda}{}_{(\mu\nu)} \;-\; \nabla_{\nu} \delta\Gamma^{\lambda}{}_{(\lambda\mu)} \tag{3.16}$$

To establish (3.16), one simply has to use the definition of the covariant derivative. What we really need is $g^{\mu\nu}\delta R_{\mu\nu}$. But since the metric is covariant constant one obtains

$$g^{\mu\nu}\delta R_{\mu\nu} \;=\; \nabla_{\lambda} \left(g^{\mu\nu} \delta\Gamma^{\lambda}{}_{(\mu\nu)} \right) \;-\; \nabla_{\nu} \left(g^{\mu\nu} \delta\Gamma^{\lambda}{}_{(\lambda\mu)} \right)$$

Since all indices are contracted we may rename them in each single term. Renaming ν to λ and vice versa one obtains

$$g^{\mu\nu}\delta R_{\mu\nu} \;=\; \nabla_{\lambda} \left(g^{\mu\nu} \delta\Gamma^{\lambda}{}_{(\mu\nu)} \;-\; g^{\mu\lambda} \delta\Gamma^{\nu}{}_{(\nu\mu)} \right) ,$$

i.e. a total divergence. The third term in (3.15) does not contribute and we obtain

$$\delta S_{(I)} \quad = \quad \int d^D x \, \sqrt{g} \, \left[\; R_{\mu\nu} \; - \; \frac{1}{2} \, g_{\mu\nu} \, R \; \right] \, \delta g^{\mu\nu} \tag{3.17}$$

Using eq. (3.14) we easily obtain the variation of the second part (II):

$$\begin{aligned} \delta S_{(II)} \quad &= \quad - \frac{1}{48} \int d^D x \, \left[\; \delta \sqrt{g} \, G_{\mu\nu\rho\sigma} \, G^{\mu\nu\rho\sigma} \; + \; \sqrt{g} \, \delta \left(G_{\mu\nu\rho\sigma} \, G^{\mu\nu\rho\sigma} \right) \; \right] \\ &= \quad - \frac{1}{48} \int d^D x \, \left[\; - \frac{1}{2} \, \sqrt{g} \, g_{\alpha\beta} \, G_{\mu\nu\rho\sigma} \, G^{\mu\nu\rho\sigma} \; + \; 4 \, \sqrt{g} \, G_{\alpha\nu\rho\sigma} \, G_{\beta}{}^{\nu\rho\sigma} \; \right] \, \delta \, g^{\alpha\beta} \end{aligned} \tag{3.18}$$

The equation of motion now reads

$$R_{\alpha\beta} \; - \; \frac{1}{2} \, g_{\alpha\beta} \, R \quad = \quad \frac{1}{48} \left(\; - \frac{1}{2} \, g_{\alpha\beta} \, G_{\mu\nu\rho\sigma} \, G^{\mu\nu\rho\sigma} \; + \; 4 \, G_{\alpha\nu\rho\sigma} \, G_{\beta}{}^{\nu\rho\sigma} \; \right) \tag{3.19}$$

Contracting the whole equation with $g^{\beta\alpha}$ we obtain

$$\frac{1}{2} \, R \quad = \quad \frac{1}{48} \left(\; \frac{1}{6} \, G_{\mu\nu\rho\sigma} \, G^{\mu\nu\rho\sigma} \; \right) \tag{3.20}$$

or

$$R_{\alpha\beta} \quad = \quad \frac{1}{12} \left(\; G_{\alpha\nu\rho\sigma} \, G_{\beta}{}^{\nu\rho\sigma} \; - \frac{1}{12} \, g_{\alpha\beta} \, G_{\mu\nu\rho\sigma} \, G^{\mu\nu\rho\sigma} \; \right) \tag{3.21}$$

3.1.3 $C_{\mu\nu\rho}$

Attention: This calculation is done in a flat metric background η_{ij}. But the modifications to a nontrivial metric are simple.

$$\mathcal{L}_C \;=\; -\frac{1}{48}\,G_{\mu\nu\rho\sigma}\,G^{\mu\nu\rho\sigma} \;+\; \frac{1}{144^2}\,\varepsilon^{\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4\gamma_1\gamma_2\gamma_3}\,G_{\alpha_1\ldots\alpha_4}\,G_{\beta_1\ldots\beta_4}\,C_{\gamma_1\gamma_2\gamma_3}$$

$$0 \;=\; \frac{\partial\mathcal{L}}{\partial\phi} \;-\; \partial_\xi\left(\frac{\partial\mathcal{L}}{\partial(\,\partial_\xi\phi\,)}\right)$$

If one chooses $\phi \,=\, C_{ijk}$ and uses the identities

$$\begin{aligned}
\frac{\partial\,G_{\mu\nu\rho\sigma}}{\partial(\,\partial_\xi C_{ijk}\,)} &= \delta^{\xi ijk}_{\mu\nu\rho\sigma} \\
\delta^{\alpha_1\ldots\alpha_4}_{\beta_1\ldots\beta_4}\,G^{\beta_1\ldots\beta_4} &= 4!\cdot G^{\alpha_1\ldots\alpha_4}
\end{aligned}$$

frequently one can compute by brute force:

$$\begin{aligned}
0 \;=\;& \frac{1}{144^2}\,\varepsilon^{\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4\gamma_1\gamma_2\gamma_3}\,G_{\alpha_1\ldots\alpha_4}\,G_{\beta_1\ldots\beta_4}\,\delta^{ijk}_{\gamma_1\gamma_2\gamma_3} \\
&- \frac{\partial}{\partial x^\xi}\Bigg[\,-\frac{1}{48}\left\{\,\delta^{\xi ijk}_{\mu\nu\rho\sigma}\,G^{\mu\nu\rho\sigma} \;+\; g^{\mu\tau_1}g^{\nu\tau_2}g^{\rho\tau_3}g^{\sigma\tau_4}G_{\mu\nu\rho\sigma}\,\delta^{\xi ijk}_{\tau_1\ldots\tau_4}\,\right\} \\
&\qquad + \frac{2}{144^2}\,\varepsilon^{\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4\gamma_1\gamma_2\gamma_3}\,\delta^{\xi ijk}_{\alpha_1\ldots\alpha_4}\,G_{\beta_1\ldots\beta_4}\,C_{\gamma_1\gamma_2\gamma_3}\,\Bigg]
\end{aligned}$$

$$\begin{aligned}
0 \;=\;& \partial_\xi G^{\xi ijk} \;+\; \frac{3!}{144^2}\,\varepsilon^{\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4 ijk}\,G_{\alpha_1\ldots\alpha_4}\,G_{\beta_1\ldots\beta_4} \\
&- \frac{2}{144^2}\,\varepsilon^{\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4\gamma_1\gamma_2\gamma_3}\,\delta^{\xi ijk}_{\alpha_1\ldots\alpha_4}\,\underbrace{G_{\beta_1\ldots\beta_4}\,\partial_\xi C_{\gamma_1\gamma_2\gamma_3}}_{\text{since } dG=0}
\end{aligned}$$

$$\begin{aligned}
0 \;=\;& \partial_\xi G^{\xi ijk} \;+\; \frac{3!}{144^2}\,\varepsilon^{\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4 ijk}\,G_{\alpha_1\ldots\alpha_4}\,G_{\beta_1\ldots\beta_4} \\
&+ \frac{2\cdot 4!}{144^2}\,\varepsilon^{\beta_1\ldots\beta_4\xi\gamma_1\gamma_2\gamma_3 ijk}\,G_{\beta_1\ldots\beta_4}\,\underbrace{\partial_\xi C_{\gamma_1\gamma_2\gamma_3}}_{\partial_{[\xi}C_{\gamma_1\gamma_2\gamma_3]}}
\end{aligned}$$

$$0 \;=\; \partial_\xi G^{\xi ijk} \;+\; \left(\frac{3!}{144^2} \;+\; \frac{2\cdot 3!}{144^2}\right)\varepsilon^{\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4 ijk}\,G_{\alpha_1\ldots\alpha_4}\,G_{\beta_1\ldots\beta_4} \tag{3.22}$$

Language of Diff. forms

To find the corresponding expression in the language of differential forms, one has to rewrite each of the terms as a 3-form (number of free indices !).

The first term must be of the form $*d*G$:

$$\begin{aligned}
G &= \frac{1}{4!}\, G_{\mu_1\mu_2\mu_3\mu_4}\, dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_4} \\
(*G)_{\mu_5\ldots\mu_{11}} &= \frac{1}{4!}\,\varepsilon_{\mu_5\ldots\mu_{11}\mu_1\ldots\mu_4}\, G^{\mu_1\ldots\mu_4} \\
d(*G)_{\nu_1\ldots\nu_8} &= 8\cdot\partial_{[\nu_1}\,\frac{1}{4!}\varepsilon_{\nu_2\ldots\nu_8]\alpha_1\ldots\alpha_4}\, G^{\alpha_1\ldots\alpha_4} \\
*d(*G)_{\nu_1\ldots\nu_3} &= \frac{1}{8!}\,\varepsilon_{\nu_1\ldots\nu_{11}}\, 8\,\partial^{[\nu_4}\,\frac{1}{4!}\varepsilon^{\nu_5\ldots\nu_{11}]\alpha_1\ldots\alpha_4}\, G_{\alpha_1\ldots\alpha_4} \\
*d(*G)_{\nu_1\ldots\nu_3} &= \frac{1}{7!\cdot 4!}\,\varepsilon_{\nu_1\ldots\nu_{11}}\,\varepsilon^{\alpha_1\ldots\alpha_4}\underbrace{{}^{[\nu_5\ldots\nu_{11}}\,\partial^{\nu_4]}}_{\nu_4\in\{\alpha_1\ldots\alpha_4\}}\, G_{\alpha_1\ldots\alpha_4} \\
*d(*G)_{\nu_1\ldots\nu_3} &= \frac{1}{7!\cdot 3!}\,\underbrace{\varepsilon_{\nu_1\ldots\nu_{11}}\,\varepsilon^{\alpha_1\ldots\alpha_3\nu_4\ldots\nu_{11}}}_{-8!\,\delta^{\alpha_1\ldots\alpha_3}_{\nu_1\ldots\nu_3}}\,\frac{1}{8}\,\partial^{\nu_4}\, G_{\alpha_1\ldots\alpha_3\nu_4} \\
&= \partial^{\xi} G_{\xi\nu_1\nu_2\nu_3}
\end{aligned}$$

The second term is related to $*(G\wedge G)$:

$$\begin{aligned}
G\wedge G &= \frac{1}{(4!)^2}\, G_{\alpha_1\ldots\alpha_4}\, G_{\beta_1\ldots\beta_4}\, dx^{\alpha_1}\wedge\ldots\wedge dx^{\alpha_4}\wedge dx^{\beta_1}\wedge\ldots\wedge dx^{\beta_4} \\
&= \frac{1}{8!}\left(\frac{8!}{(4!)^2}\, G_{\alpha_1\ldots\alpha_4}\, G_{\beta_1\ldots\beta_4}\right) dx^{\alpha_1}\wedge\ldots\wedge dx^{\alpha_4}\wedge dx^{\beta_1}\wedge\ldots\wedge dx^{\beta_4} \\
\left[*(G\wedge G)\right]_{\nu_1\nu_2\nu_3} &= \frac{1}{8!}\,\varepsilon_{\nu_1\ldots\nu_3\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4}\,\frac{8!}{(4!)^2}\, G^{\alpha_1\ldots\alpha_4}\, G^{\beta_1\ldots\beta_4} \\
\left[*(G\wedge G)\right]_{\nu_1\nu_2\nu_3} &= \frac{1}{(4!)^2}\,\varepsilon_{\nu_1\ldots\nu_3\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4}\, G^{\alpha_1\ldots\alpha_4}\, G^{\beta_1\ldots\beta_4}
\end{aligned}$$

Using this both coordinate expressions we can rewrite eq. (3.22) as follows:

$$\begin{aligned}
0 &= *d(*G) + (4!)^2\left(\frac{3!}{144^2} + \frac{2\cdot 3!}{144^2}\right) *(G\wedge G) \\
0 &= d(*G) + \frac{1}{2}\, G\wedge G \qquad (3.23)
\end{aligned}$$

Homework:

Exercise 3. *Compute the Ricci tensor of the following diagonal metric in D dimensions:*

$$g_{\mu\nu} = N\cdot\delta_{\mu\nu}$$

4 Constructing the Lagrangian

In Chapter 3 the Lagrangian of eleven dimensional supergravity was introduced. Now we are going to discuss its structure in more detail by reviewing some of the basic steps of its construction along the lines of the original paper [2]. We performing the construction of the complete bosonic action, which requires the construction up to second order of the fermionic terms in the action. The starting point is an action where we simply added together an Einstein-Hilbert describing pure gravity with a Rarita-Schwinger action describing a Majorana gravitino, two of the three fields forming the representation of $N = 1$ supersymmetry in $D = 11$:

$$S \quad = \quad \int dx^D \sqrt{g} \left[\frac{1}{4} R + \frac{1}{2} \bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu (\omega) \psi_\rho \right] . \tag{4.1}$$

An ansatz for the linearised transformation law of supersymmetry is:

$$\begin{aligned} \delta_Q \psi_\mu &= D_\mu(\omega)\epsilon \\ \delta_Q e_\mu{}^a &= \bar{\epsilon}\,\Gamma^a \psi_\mu \quad \Rightarrow \quad \delta_Q g_{\mu\nu} \; = \; \delta_Q \left(\eta_{ab}\, e^a_\mu\, e^b_\nu \right) \; = \; 2\,\bar{\epsilon}\,\Gamma_\mu \psi_\nu \, . \end{aligned} \tag{4.2}$$

Performing the variation of the action according to a symmetry δ_Q and using the result eq. (3.17) one obtains

$$\begin{aligned} \delta_Q S \quad = \quad \int dx^D \sqrt{g} \Bigg[& \frac{1}{4} \left(R_{\mu\nu} \; - \; \frac{1}{2} g_{\mu\nu} R \right) \delta_Q g^{\mu\nu} \\ & + \underbrace{\frac{1}{2}\, \delta_Q \bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu (\omega)\, \psi_\rho}_{(2\text{nd})} - \underbrace{\frac{1}{2}\, \bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu (\omega)\, \delta_Q \psi_\rho}_{(3\text{th})} \\ & + \underbrace{\frac{3}{2}\, \bar{\psi}_{[\mu} \delta_Q g^{\mu\alpha} \Gamma_\alpha{}^{\nu\rho} D_\nu (\omega)\, \psi_{\rho]}}_{(4\text{th})} \Bigg] \end{aligned}$$

The fourth term is of third order in the fermionic fields and must be cancelled by other terms containing three fermions yet to be added. Therefore, on this level, i.e. $\mathcal{O}(\bar{\epsilon}\psi G^0)$, it can be neglected. Now we proceed to simplify the second term in (4.1), i.e.

$$(2\text{nd}) \quad = \quad \frac{1}{2}\, \overline{D_\mu \epsilon}\, \Gamma^{\mu\nu\rho} D_\nu (\omega)\, \psi_\rho \, . \tag{4.3}$$

By partial integration it reads

$$(2\text{nd}) \quad = \quad -\frac{1}{2}\, \bar{\epsilon}\, \Gamma^{\mu\nu\rho} D_\mu\, D_\nu (\omega)\, \psi_\rho \, . \tag{4.4}$$

Due to

$$D_{[\mu}D_{\nu]}\epsilon \quad = \quad \frac{1}{8}\,R_{\mu\nu}{}^{\alpha\beta}\,\Gamma_{\alpha\beta}\,\epsilon \tag{4.5}$$

we obtain

$$(2\mathrm{nd}) \quad = \quad -\frac{1}{16}\,R_{\mu\nu}{}^{\alpha\beta}\,\bar\epsilon\,\Gamma^{\mu\nu\rho}\,\Gamma_{\alpha\beta}\psi_\rho\ . \tag{4.6}$$

The right hand side can be expanded in the Clifford algebra and gives:

$$\begin{aligned} R_{\mu\nu}{}^{\alpha\beta}\,\bar\epsilon\,\Gamma^{\mu\nu\rho}\Gamma_{\alpha\beta}\,\psi_\rho \quad &= \quad R_{\mu\nu}{}^{\alpha\beta}\,\bar\epsilon\Big\{\ \Gamma^{\mu\nu\rho}{}_{\alpha\beta} + 6\,\delta^{[\rho}_{[\alpha}\Gamma^{\mu\nu]}{}_{\beta]} + 6\,\delta^{[\rho}_{[\alpha}\delta^{\nu}_{\beta]}\Gamma^{\mu]}\ \Big\}\,\psi_\rho \\ &= \quad R_{\mu\nu}{}^{\alpha\beta}\,\bar\epsilon\Big\{\ \Gamma^{\mu\nu\rho}{}_{\alpha\beta} + 6\,\delta^{[\rho}_{[\alpha}\Gamma^{\mu\nu]}{}_{\beta]}\ \Big\}\,\psi_\rho\ + \\ &\qquad R_{\mu\nu}{}^{\alpha\beta}\,\bar\epsilon\Big\{\ \delta^\rho_\alpha\delta^\nu_\beta\Gamma^\mu - \delta^\rho_\alpha\delta^\mu_\beta\Gamma^\nu + \delta^\mu_\alpha\delta^\rho_\beta\Gamma^\nu \\ &\qquad\qquad -\delta^\mu_\alpha\delta^\nu_\beta\Gamma^\rho + \delta^\nu_\alpha\delta^\mu_\beta\Gamma^\rho - \delta^\nu_\alpha\delta^\rho_\beta\Gamma^\mu\ \Big\}\,\psi_\rho \\ &= \quad R_{\mu\nu}{}^{\alpha\beta}\,\bar\epsilon\Big\{\ \Gamma^{\mu\nu\rho}{}_{\alpha\beta} + 6\,\delta^{[\rho}_{[\alpha}\Gamma^{\mu\nu]}{}_{\beta]}\ \Big\}\,\psi_\rho\ +\ \bar\epsilon\Big\{\ 4\,R_\mu{}^\rho\,\Gamma^\mu\ - 2\,R\,\Gamma^\rho\ \Big\}\,\psi_\rho \\ &= \quad R_{\mu\nu}{}^{\alpha\beta}\,\bar\epsilon\Big\{\ \Gamma^{\mu\nu\rho}{}_{\alpha\beta} + 6\,\delta^{[\rho}_{[\alpha}\Gamma^{\mu\nu]}{}_{\beta]}\ \Big\}\,\psi_\rho\ +\ 4\,\Big\{\ R^{\mu\rho}\ -\ \frac{1}{2}\,R\,g^{\mu\rho}\ \Big\}\,\bar\epsilon\,\Gamma_\mu\,\psi_\rho\ . \end{aligned}$$

The second term finally reads

$$(2\mathrm{nd}) \quad = \quad -\frac{1}{16}\,R_{\mu\nu}{}^{\alpha\beta}\,\bar\epsilon\Big\{\ \Gamma^{\mu\nu\rho}{}_{\alpha\beta} + 6\,\delta^{[\rho}_{[\alpha}\Gamma^{\mu\nu]}{}_{\beta]}\ \Big\}\,\psi_\rho\ -\ \frac{1}{4}\,\Big\{\ R^{\mu\rho}\ -\ \frac{1}{2}\,R\,g^{\mu\rho}\ \Big\}\,\bar\epsilon\,\Gamma_\mu\,\psi_\rho\ . \tag{4.7}$$

The same steps must be applied to the third term and we obtain:

$$(3\mathrm{th}) \quad = \quad \frac{1}{16}\,R_{\nu\rho}{}^{\alpha\beta}\,\bar\psi_\mu\Big\{\ \Gamma^{\mu\nu\rho}{}_{\alpha\beta} + 6\,\delta^{[\rho}_{[\alpha}\Gamma^{\mu\nu]}{}_{\beta]}\ \Big\}\,\epsilon\ +\ \frac{1}{4}\,\Big\{\ R^{\rho\mu}\ -\ \frac{1}{2}\,R\,g^{\mu\rho}\ \Big\}\,\bar\psi_\mu\,\Gamma_\rho\,\epsilon\ . \tag{4.8}$$

To compare the second with the third term in the latter one one has to switch the position of ϵ and ψ via the identity (2.71). Then the term with $\Gamma^{(3)}$ picks up a sign. Subtracting the third from the second term we end up with:

$$
\begin{aligned}
(\text{2nd}) \; - \; (\text{3th}) \quad = \quad & \frac{1}{16}\,(\,-1\;-\;1\,)\,R_{\mu\nu\alpha\beta}\,\bar{\epsilon}\,\Gamma^{\mu\nu\rho\alpha\beta}\,\psi_\rho \\
& -\frac{6}{16}\,(\quad 1\;-\;1\,)\,R_{\mu\nu}{}^{\alpha\beta}\,\bar{\epsilon}\,\delta^{[\rho}_{[\alpha}\Gamma^{\mu\nu]}{}_{\beta]}\,\psi_\rho \\
& +\frac{1}{4}\,(\,-1\;-\;1\,)\,\left\{\; R^{\mu\rho} \;-\; \frac{1}{2}\,R\,g^{\mu\rho} \;\right\}\,\bar{\epsilon}\,\Gamma_\mu\,\psi_\rho \;.
\end{aligned}
$$

Obviously, on the right hand side the second line vanishes and the first line does too, due to a symmetry of the curvature tensor ($R_{\mu[\nu\alpha\beta]} \;=\; 0$):

$$
R_{\mu\nu\alpha\beta}\,\Gamma^{\mu\nu\rho\alpha\beta} \quad = \quad \frac{1}{3}\,\Big(\,\underbrace{R_{\mu\nu\alpha\beta} \,+ R_{\mu\beta\nu\alpha} \,+\, R_{\mu\alpha\beta\nu}}_{0}\,\Big)\,\Gamma^{\mu\nu\rho\alpha\beta}\;. \tag{4.9}
$$

Plugging these results back into the variation of the action it reads

$$
\delta_Q S \quad = \quad \int dx^D\,\sqrt{g}\,\left[\;\frac{1}{2}\,\left(\;R^{\mu\nu} \;-\; \frac{1}{2}\,g^{\mu\nu}\,R\;\right)\,\bar{\epsilon}\,\Gamma_{(\mu}\psi_{\nu)} \;-\; \frac{1}{2}\,\left(\;R^{\mu\nu} \;-\; \frac{1}{2}\,g^{\mu\nu}\,R\;\right)\,\bar{\epsilon}\,\Gamma_{(\mu}\,\psi_{\nu)}\;\right]\;.
$$

Here we added the two symmetrisation symbols on μ and ν, because the contraction with the symmetric Einstein tensor picks out the symmetric combination. To this order the variation of the action vanishes identically. Up to here and to the chosen order $\mathcal{O}(\bar{\epsilon}\,\psi\,G^0)$ the action (4.1) and the supersymmetry variation (4.2) are consistent. Now we add in the kinetic term for the three form potential, i.e.

$$
S \quad = \quad \int dx^D\,\sqrt{g}\,\left[\frac{1}{4}\,R + \frac{1}{2}\,\bar{\psi}_\mu\Gamma^{\mu\nu\rho}D_\nu\,(\omega)\,\psi_\rho \;-\; \frac{1}{4\cdot 48}\,G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma}\right]\;. \tag{4.10}
$$

On the same level of the fermionic fields considered before the variation of the additional term with respect to $g^{\mu\nu}$ produces now a new contribution of $-1/24\,\bar{\epsilon}\,\Gamma^\mu\psi^\nu\,(G^2)_{\mu\nu}$, which must be balanced by the following modification of the action

$$
S \quad = \quad \int dx^D\,\sqrt{g}\,\left[\frac{1}{4}\,R + \frac{1}{2}\,\bar{\psi}_\mu\Gamma^{\mu\nu\rho}D_\nu\,(\omega)\,\psi_\rho + \bar{\psi}_\mu(XG)^{\mu\rho}\,\psi_\rho \;-\; \frac{1}{4\cdot 48}\,G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma}\right] \tag{4.11}
$$

and subsequently by the following modified supersymmetry transformations

$$
\begin{aligned}
\delta_Q g_{\mu\nu} \quad &= \quad 2\,\bar{\epsilon}\,\Gamma_\mu\psi_\nu \\
\delta_Q \psi_\mu \quad &= \quad D_\mu(\omega)\epsilon + (ZG)_\mu\epsilon \;\;=\;\; \hat{D}_\mu(\omega)\epsilon\;.
\end{aligned} \tag{4.12}
$$

The actual determination of $(XG)^{\mu\nu}$ is a bit messy but straightforward and is in fact a generalisation

of the calculations done to prove Proposition 7 in subsection 3.3. The logic is first to write down all possible terms consistent with the tensor structure $(XG)^{\mu\nu}$. The building blocks are the fields and Gamma matrices. There are just five possibilities to write a tensor $\bar{\psi}_\mu (XG)^{\mu\rho}\,\psi_\rho$ in the Clifford algebra. Depending on the symmetry in μ and ν one obtains:

$$\begin{aligned}(XG)^{\mu\nu} &= \underbrace{a\,\Gamma^{(\mu}{}_{\alpha_1\ldots\alpha_3}\,G^{\nu)\alpha_1\ldots\alpha_3} + b\,g^{\mu\nu}\,\Gamma_{\alpha_1\ldots\alpha_4}G^{\alpha_1\ldots\alpha_4}}_{\text{symm.}} \\ &+ \underbrace{c\,\Gamma^{\mu\nu}{}_{\alpha_1\ldots\alpha_4}\,G^{\alpha_1\ldots\alpha_4} + d\,\Gamma_{\alpha_1\alpha_2}\,G^{\mu\nu\alpha_1\alpha_2} + e\,\Gamma^{[\mu}{}_{\alpha_1\alpha_2\alpha_3}\,G^{\nu]\alpha_1\alpha_2\alpha_3}}_{\text{antisymm.}}\ .\end{aligned}$$

When we derived the equation of motion of the gravitino (3.9), we observed its close relationship to the operator governing the supersymmetry variation of the gravitino. This observation can be utilised to determine $(ZG)_\mu$ directly. Just compute the equation of motion of the gravitino and read of the definition of $(ZG)_\mu$, again. Now $(ZG)_\mu$ also becomes a tensor in the five real unknowns a, b, c, d, e. Using the so determined tensors one has to perform a supervariation of the action in the appropriate level of fermions and obtains a set of equations to fix the five unknowns. The computation one has to do is straightforward but tedious. To cut the story short it finally turns out that only two antisymmetric terms occur with $c = -(8\cdot 4!)^{-1}$ and $d = -12\cdot(8\cdot 4!)^{-1}$. According to Proposition 7 on page 34 the modification of the supersymmetry variation is then given by

$$(ZG)_\mu \quad = \quad -\frac{1}{2\cdot 144}\left(\Gamma^{\alpha\beta\gamma\delta}{}_\mu - 8\,\Gamma^{\beta\gamma\delta}\delta^\alpha_\mu\right)\,G_{\alpha\beta\gamma\delta}\ . \tag{4.13}$$

Plan: Now we want to recompute $\delta_Q S$ up to order $\mathcal{O}(\bar{\epsilon}\psi)G^2$ in order to check if the action (4.11) constructed so far is consistent. We will discover the need to add in an additional term, the so called Chern-Simons-Term.

The terms in the variation of the action (4.11) of order $\mathcal{O}(\bar{\epsilon}\psi)G^2$ are generated from the two pieces

$$\delta_Q S \quad = \quad \int dx^D\,\Big[\,\underbrace{\delta_Q\Big(-\frac{e}{4\cdot 48}\,\bar{\psi}_\mu (XG)^{\mu\rho}\,\psi_\rho\Big)}_{(II)} \;+\; \underbrace{\delta_Q\Big(-\frac{e}{4\cdot 48}\,G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma}\Big)}_{(I)}\,\Big]\ . \tag{4.14}$$

Using (3.14) and (4.12) one obtains

$$\begin{aligned}(I) \quad &= \quad \delta\left(-\frac{e}{4\cdot 48}\,G_{\alpha_1\ldots\alpha_4}\,g^{\alpha_1\beta_1}\cdots g^{\alpha_4\beta_4}\,G_{\beta_1\ldots\beta_4}\right) \\ &\overset{\bar{\epsilon}\psi G^2}{=} \quad -\frac{1}{4\cdot 48}\,G^2\cdot\left(-\frac{1}{2}\,e\,g_{jk}\,2\,\bar{\epsilon}\,\Gamma^{(j}\,\psi^{k)}\right) - \frac{4\cdot e}{4\cdot 48}\,(G^2)_{\alpha_1\beta_1}\,2\,\bar{\epsilon}\,\Gamma^{(\alpha_1}\,\psi^{\beta_1)} \\ &= \quad -\frac{e}{24}\left((G^2)_{\alpha_1\beta_1} - \frac{1}{8}\,G^2\cdot g_{\alpha_1\beta_1}\right)\bar{\epsilon}\,\Gamma^{(\alpha_1}\,\psi^{\beta_1)}\ .\end{aligned} \tag{4.15}$$

The evaluation of the second piece is more complicated:

$$
\begin{aligned}
(II) \quad &= \quad \delta \left(- \frac{e}{4 \cdot 48} \left(\bar{\psi}_\mu \, \Gamma^{\mu\nu\alpha\beta\gamma\delta} \psi_\nu \, + \, 12 \, \bar{\psi}^\alpha \, \Gamma^{\gamma\delta} \psi^\beta \right) G_{\alpha\beta\gamma\delta} \right) \\
&\overset{\bar{\epsilon}\psi G^2}{=} \quad - \frac{e}{4 \cdot 48} \, \delta \left(\bar{\psi}_\mu \, \Gamma^{\mu\nu\alpha\beta\gamma\delta} \psi_\nu \, + \, 12 \, \bar{\psi}^\alpha \, \Gamma^{\gamma\delta} \psi^\beta \right) G_{\alpha\beta\gamma\delta}
\end{aligned}
\tag{4.16}
$$

The first problem is to determine the variation of $\bar{\psi}$.

1st Auxiliary calculation:

$$
\begin{aligned}
\delta \bar{\psi}_\mu \quad &= \quad \delta \left(\psi_\mu^T \, \Gamma^0 \right) \quad = \quad \left(\delta \psi_\mu \right)^T \, \Gamma^0 \\
&= \quad \left(D_\mu \epsilon - \frac{1}{2 \cdot 144} \left[\Gamma^{\alpha\beta\gamma\delta}{}_\mu \, - \, 8 \, \Gamma^{\beta\gamma\delta} \, \delta_\mu^\alpha \right] G_{\alpha\beta\gamma\delta} \, \epsilon \right)^T \, \Gamma^0 \\
&= \quad \left(D_\mu \epsilon \right)^T \, \Gamma^0 \, - \, \frac{1}{2 \cdot 144} \left(\overline{\Gamma^{\alpha\beta\gamma\delta}{}_\mu \, \epsilon} \, - \, 8 \, \overline{\Gamma^{\beta\gamma\delta} \, \epsilon} \, \delta_\mu^\alpha \right) G_{\alpha\beta\gamma\delta}
\end{aligned}
$$

2nd Auxiliary calculation:

$$
\begin{aligned}
\overline{\Gamma^{(j)} \, \epsilon} \quad &= \quad \left(\Gamma^{(j)} \, \epsilon \right)^T \, \Gamma^0 \\
&= \quad \epsilon^T \, (\Gamma^{a_j})^T \ldots (\Gamma^{a_1})^T \, \Gamma^0 \\
&= \quad (-1)^j \, \epsilon^T \, (\Gamma^0)^2 \, (\Gamma^{a_j})^T \, (\Gamma^0)^2 \ldots \, (\Gamma^0)^2 \, (\Gamma^{a_1})^T \, \Gamma^0 \\
&= \quad (-1)^j \, \bar{\epsilon} \, \Gamma^{a_j} \ldots \Gamma^{a_1} \\
&= \quad (-1)^j \, (-1)^{\frac{(j-1)j}{2}} \, \bar{\epsilon} \, \Gamma^{a_1} \ldots \Gamma^{a_j} \\
&= \quad (-1)^{\frac{(j+1)j}{2}} \, \bar{\epsilon} \, \Gamma^{(j)}
\end{aligned}
$$

$$
\Rightarrow \quad \delta \bar{\psi}_\mu \quad = \quad \left(D_\mu \epsilon \right)^T \, \Gamma^0 \, + \, \frac{1}{2 \cdot 144} \, \bar{\epsilon} \left(\Gamma^{\alpha\beta\gamma\delta}{}_\mu \, + \, 8 \, \Gamma^{\beta\gamma\delta} \, \delta_\mu^\alpha \right) G_{\alpha\beta\gamma\delta}
$$

Inserting the variation of ψ and $\bar{\psi}$ and keeping in mind the sign from anticommuting the supervariation with the spinor one obtains:

$$
\begin{aligned}
(II) \quad &= \quad - \frac{e}{4 \cdot 48} \left\{ \delta \bar{\psi}_\mu \left[\Gamma^{\mu\nu\alpha\beta\gamma\delta} \psi_\nu \, + \, 12 \, g^{\mu[\alpha} \, \Gamma^{\gamma\delta} \psi^{\beta]} \right] \, - \, \left[\bar{\psi}_\mu \Gamma^{\mu\nu\alpha\beta\gamma\delta} \, + \, 12 \, \bar{\psi}^{[\alpha} \, \Gamma^{\gamma\delta} g^{\beta]\nu} \right] \delta \psi_\nu \right\} G_{\alpha\beta\gamma\delta} \\
&\overset{\bar{\epsilon}\psi G^2}{=} \quad - \frac{e}{32 \cdot (12)^3} \left\{ \, \bar{\epsilon} \left(\Gamma^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}}{}_\mu \, + \, 8 \, \Gamma^{\tilde{\beta}\tilde{\gamma}\tilde{\delta}} \, \delta_\mu^{\tilde{\alpha}} \right) \left[\Gamma^{\mu\nu\alpha\beta\gamma\delta} \, + \, 12 \, g^{\mu[\alpha} \, \Gamma^{\gamma\delta} g^{\beta]\nu} \right] \psi_\nu \right. \\
&\qquad \left. - \, \bar{\psi}_\nu \left[\Gamma^{\mu\nu\alpha\beta\gamma\delta} \, - \, 12 \, g^{\nu[\alpha} \, \Gamma^{\gamma\delta} g^{\beta]\mu} \right] \left(\Gamma^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}}{}_\mu \, - \, 8 \, \Gamma^{\tilde{\beta}\tilde{\gamma}\tilde{\delta}} \, \delta_\mu^{\tilde{\alpha}} \right) \epsilon \, \right\} G_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}} \, G_{\alpha\beta\gamma\delta}
\end{aligned}
$$

In section 3.1 we proved formula (3.8). Performing a simple transposition and reordering the Gamma matrices one obtains the identity (exercise 4)

$$(\Gamma_{\alpha\beta\gamma\delta\nu} + 8\,\Gamma_{\beta\gamma\delta}\,\eta_{\nu\alpha})\;\Gamma^{\mu\nu\rho}\,\hat{G}^{\alpha\beta\gamma\delta} \;\;=\;\; 3\cdot\big(\;\Gamma^{\mu\rho}{}_{\alpha\beta\gamma\delta} \,+\, 12\,\delta^{\mu}_{\alpha}\,\Gamma_{\gamma\delta}\,\delta^{\rho}_{\beta}\;\big)\;\hat{G}^{\alpha\beta\gamma\delta}\;. \tag{4.17}$$

Inserting (3.8) and (4.17) into II one obtains:

$$\begin{aligned}(II) \;\;=\;\; -\frac{e}{3\cdot 32\cdot(12)^3}\,\Big\{\;&\bar{\epsilon}\,\Big(\Gamma^{\tilde\alpha\tilde\beta\tilde\gamma\tilde\delta}{}_{\mu} + 8\,\Gamma^{\tilde\beta\tilde\gamma\tilde\delta}\,\delta^{\tilde\alpha}_{\mu}\Big)\,\Gamma^{\mu\rho\nu}\,\Big(\Gamma^{\tilde\alpha\tilde\beta\tilde\gamma\tilde\delta}{}_{\rho} - 8\,\Gamma^{\tilde\beta\tilde\gamma\tilde\delta}\,\delta^{\tilde\alpha}_{\rho}\Big)\,\psi_\nu \\ &+\;\bar{\psi}_\nu\,\Big(\Gamma^{\tilde\alpha\tilde\beta\tilde\gamma\tilde\delta}{}_{\rho} + 8\,\Gamma^{\tilde\beta\tilde\gamma\tilde\delta}\,\delta^{\tilde\alpha}_{\rho}\Big)\,\Gamma^{\nu\rho\mu}\,\Big(\Gamma^{\tilde\alpha\tilde\beta\tilde\gamma\tilde\delta}{}_{\mu} - 8\,\Gamma^{\tilde\beta\tilde\gamma\tilde\delta}\,\delta^{\tilde\alpha}_{\mu}\Big)\,\epsilon\;\Big\}\;G_{\tilde\alpha\tilde\beta\tilde\gamma\tilde\delta}\,G_{\alpha\beta\gamma\delta}\end{aligned}$$

The advantage of this representation of (II) is that the symmetry is made explicit. In exercise 1 we developed formula (2.71), which tells us now, that out of the above expressions just the Gamma matrices with 1,2 and 5 antisymmetrised indices and the corresponding Hodge duals (6,9,10) survive. Keeping just the first line, the corresponding terms are doubled:

$$(II) \;\;=\;\; -\underbrace{\frac{2\cdot e}{3\cdot 32\cdot(12)^3}}_{e\,\frac{1}{4\cdot(12)^4}}\;\Big\{\;\bar{\epsilon}\,\Big(\Gamma^{\tilde\alpha\tilde\beta\tilde\gamma\tilde\delta}{}_{\mu} + 8\,\Gamma^{\tilde\beta\tilde\gamma\tilde\delta}\,\delta^{\tilde\alpha}_{\mu}\Big)\,\Gamma^{\mu\rho\nu}\,\Big(\Gamma^{\tilde\alpha\tilde\beta\tilde\gamma\tilde\delta}{}_{\rho} - 8\,\Gamma^{\tilde\beta\tilde\gamma\tilde\delta}\,\delta^{\tilde\alpha}_{\rho}\Big)\,\psi_\nu\;\Big\}_{1,2,5}\;G_{\tilde\alpha\tilde\beta\tilde\gamma\tilde\delta}\,G_{\alpha\beta\gamma\delta}$$

Using eq. (3.8) ones more we obtain again

$$\begin{aligned}(II) \;\;&=\;\; -\frac{3\,e}{4\cdot(12)^4}\,\Big\{\;\bar{\epsilon}\,\Big(\Gamma^{\tilde\alpha\tilde\beta\tilde\gamma\tilde\delta}{}_{\mu} + 8\,\Gamma^{\tilde\beta\tilde\gamma\tilde\delta}\,\delta^{\tilde\alpha}_{\mu}\Big)\,\Big[\Gamma^{\mu\nu\alpha\beta\gamma\delta} + 12\,g^{\mu[\alpha}\,\Gamma^{\gamma\delta}g^{\beta]\nu}\Big]\;\psi_\nu\;\Big\}_{1,2,5}\;G_{\tilde\alpha\tilde\beta\tilde\gamma\tilde\delta}\,G_{\alpha\beta\gamma\delta} \\ &=\;\; -\frac{3\,e}{4\cdot(12)^4}\,\Big\{\;\bar{\epsilon}\,\Gamma^{\tilde\alpha\tilde\beta\tilde\gamma\tilde\delta}{}_{\mu}\,\Gamma^{\mu\nu\alpha\beta\gamma\delta}\,\psi_\nu + 12\,\bar{\epsilon}\,\Gamma^{\tilde\alpha\tilde\beta\tilde\gamma\tilde\delta}{}_{\mu}\,g^{\mu[\alpha}\,\Gamma^{\gamma\delta}\,g^{\beta]\nu}\,\psi_\nu \\ &\qquad\qquad + 8\,\bar{\epsilon}\,\Gamma^{\tilde\beta\tilde\gamma\tilde\delta}\,\delta^{\tilde\alpha}_{\mu}\,\Gamma^{\mu\nu\alpha\beta\gamma\delta}\,\psi_\nu + 96\,\bar{\epsilon}\,\Gamma^{\tilde\beta\tilde\gamma\tilde\delta}\,\delta^{\tilde\alpha}_{\mu}\,g^{\mu[\alpha}\,\Gamma^{\gamma\delta}\,g^{\beta]\nu}\,\psi_\nu\;\Big\}_{1,2,5}\;G_{\tilde\alpha\tilde\beta\tilde\gamma\tilde\delta}\,G_{\alpha\beta\gamma\delta}\end{aligned}$$

To further simplify this expression one needs the following identity,

$$\Gamma^{\tilde\alpha\tilde\beta\tilde\gamma\tilde\delta}{}_{\mu}\Gamma^{\mu\nu\alpha\beta\gamma\delta} \;\;=\;\; (D-9)\,\Gamma^{\tilde\alpha\tilde\beta\tilde\gamma\tilde\delta}\,\Gamma^{\nu\alpha\beta\gamma\delta} + 20\,\Gamma^{[\tilde\alpha\tilde\beta\tilde\gamma}\,g^{\tilde\delta][\nu}\Gamma^{\alpha\beta\gamma\delta]}\;, \tag{4.18}$$

the proof of which is left as an exercise. One obtains

$$(II) \quad = \quad -\frac{3\,e}{4\cdot(12)^4}\,\Big\{\ (D-9)\,\bar\epsilon\,\Gamma^{\tilde\alpha\tilde\beta\tilde\gamma\tilde\delta}\,\Gamma^{\nu\alpha\beta\gamma\delta}\,\psi_\nu \;+\; 20\,\bar\epsilon\,\Gamma^{[\tilde\alpha\tilde\beta\tilde\gamma}g^{\tilde\delta][\nu}\,\Gamma^{\alpha\beta\gamma\delta]}\,\psi_\nu$$

$$+\,12\,\bar\epsilon\,\Gamma^{\tilde\alpha\tilde\beta\tilde\gamma\tilde\delta}{}_\mu\,g^{\mu[\alpha}\,\Gamma^{\gamma\delta}\,g^{\beta]\nu}\,\psi_\nu \;+\; 8\,\bar\epsilon\,\Gamma^{\tilde\beta\tilde\gamma\tilde\delta}\,\delta^{\tilde\alpha}_\mu\,\Gamma^{\mu\nu\alpha\beta\gamma\delta}\,\psi_\nu$$

$$+\,96\,\bar\epsilon\,\Gamma^{\tilde\beta\tilde\gamma\tilde\delta}\,\delta^{\tilde\alpha}_\mu\,g^{\mu[\alpha}\,\Gamma^{\gamma\delta}\,g^{\beta]\nu}\,\psi_\nu\ \Big\}_{1,2,5}\,G_{\tilde\alpha\tilde\beta\tilde\gamma\tilde\delta}\,G_{\alpha\beta\gamma\delta}$$

Another useful identity is

$$8\,\bar\epsilon\,\Gamma^{\tilde\beta\tilde\gamma\tilde\delta}\,\delta^{\tilde\alpha}_\mu\,\Gamma^{\mu\nu\alpha\beta\gamma\delta}\,\psi_\nu \quad = \quad 8\,\bar\epsilon\,\Gamma^{\tilde\beta\tilde\gamma\tilde\delta\tilde\alpha}\,\Gamma^{\nu\alpha\beta\gamma\delta}\,\psi_\nu \;+\; 40\,\bar\epsilon\,\Gamma^{[\tilde\alpha\tilde\beta\tilde\gamma}\,g^{\tilde\delta][\nu}\,\Gamma^{\alpha\beta\gamma\delta]}\,\psi_\nu \tag{4.19}$$

and inserting this one obtains

$$(II) \quad = \quad -\frac{3\,e}{4\cdot(12)^4}\,\Big\{\ (D-9-8)\,\bar\epsilon\,\Gamma^{\tilde\alpha\tilde\beta\tilde\gamma\tilde\delta}\,\Gamma^{\nu\alpha\beta\gamma\delta}\,\psi_\nu \;+\; 60\,\bar\epsilon\,\Gamma^{[\tilde\alpha\tilde\beta\tilde\gamma}g^{\tilde\delta][\nu}\,\Gamma^{\alpha\beta\gamma\delta]}\,\psi_\nu$$

$$+\,12\,\bar\epsilon\,\Gamma^{\tilde\alpha\tilde\beta\tilde\gamma\tilde\delta}{}_\mu\,g^{\mu[\alpha}\,\Gamma^{\gamma\delta}\,g^{\beta]\nu}\,\psi_\nu \;+\; 96\,\bar\epsilon\,\Gamma^{\tilde\beta\tilde\gamma\tilde\delta}\,\delta^{\tilde\alpha}_\mu\,g^{\mu[\alpha}\,\Gamma^{\gamma\delta}\,g^{\beta]\nu}\,\psi_\nu\ \Big\}_{1,2,5}\,G_{\tilde\alpha\tilde\beta\tilde\gamma\tilde\delta}\,G_{\alpha\beta\gamma\delta}$$

The remaining four terms must be expanded in a Clifford basis by brute force. The result is

$$\begin{aligned}
(II) \quad = \quad -\frac{3\,e}{4\cdot(12)^4}\,\Big\{ & \\
& (D-9-8)\,\bar\epsilon\,\Big[\ \Gamma_{\tilde\alpha\tilde\beta\tilde\gamma\tilde\delta}{}^{\nu\alpha\beta\gamma\delta} \;+\; 20\,\delta^{[\nu}_{[\tilde\delta}\Gamma_{\tilde\alpha\tilde\beta\tilde\gamma]}{}^{\alpha\beta\gamma\delta]} \;+\; 120\,\delta^{[\nu}_{[\tilde\delta}\delta^{\alpha}_{\tilde\gamma}\Gamma_{\tilde\alpha\tilde\beta]}{}^{\beta\gamma\delta]} \\
& \qquad +\,240\,\delta^{[\nu}_{[\tilde\delta}\delta^{\alpha}_{\tilde\gamma}\delta^{\beta}_{\tilde\beta}\Gamma_{\tilde\alpha]}{}^{\gamma\delta]} \;+\; 120\,\delta^{[\nu}_{[\tilde\delta}\delta^{\alpha}_{\tilde\gamma}\delta^{\beta}_{\tilde\beta}\delta^{\gamma}_{\tilde\alpha]}\Gamma^{\delta]}\ \Big]\ \psi_\nu \\
& +\,60\,\bar\epsilon\,\delta^{[\nu}_{\tilde\delta}\,\Big[\ \Gamma_{\tilde\alpha\tilde\beta\tilde\gamma}{}^{\alpha\beta\gamma\delta]} \;+\; 12\,\delta^{\alpha}_{\tilde\gamma}\Gamma_{\tilde\alpha\tilde\beta}{}^{\beta\gamma\delta]} \;+\; 36\,\delta^{\alpha}_{\tilde\gamma}\delta^{\beta}_{\tilde\beta}\Gamma_{\tilde\alpha}{}^{\gamma\delta]} \;+\; 24\,\delta^{\alpha}_{\tilde\gamma}\delta^{\beta}_{\tilde\beta}\delta^{\gamma}_{\tilde\alpha}\Gamma^{\delta]}\ \Big]\ \psi_\nu \\
& +\,12\,\bar\epsilon\,g^{\mu\alpha}\,\Big[\ \Gamma_{\tilde\alpha\tilde\beta\tilde\gamma\tilde\delta\mu}{}^{\gamma\delta} \;+\; 10\,\delta^{\gamma}_{[\mu}\Gamma_{\tilde\alpha\tilde\beta\tilde\gamma\tilde\delta]}{}^{\delta} \;+\; 20\,\delta^{\gamma}_{[\mu}\delta^{\delta}_{\tilde\delta}\Gamma_{\tilde\alpha\tilde\beta\tilde\gamma]}\ \Big]\ g^{\beta\nu}\,\psi_\nu \\
& +\,96\,\bar\epsilon\,\delta^{\alpha}_{\tilde\alpha}\,\Big[\ \Gamma_{\tilde\beta\tilde\gamma\tilde\delta}{}^{\gamma\delta} \;+\; 6\,\delta^{\gamma}_{\tilde\delta}\Gamma_{\tilde\beta\tilde\gamma}{}^{\delta} \;+\; 6\,\delta^{\gamma}_{\tilde\delta}\delta^{\delta}_{\tilde\gamma}\Gamma_{\tilde\beta}\ \Big]\ \psi^\beta \\
& \Big\}_{1,2,5}\,G_{\tilde\alpha\tilde\beta\tilde\gamma\tilde\delta}\,G^{\alpha\beta\gamma\delta}
\end{aligned} \tag{4.20}$$

Collecting the terms in (4.20) with just one Gamma matrix one finds

$$
\begin{aligned}
\bar{\epsilon}\Gamma^{(1)}\psi &= -\frac{3}{4\cdot(12)^4}\left\{ -\,6\cdot 120\,\bar{\epsilon}\,\delta^{[\alpha}_{\tilde{\delta}}\delta^{\beta}_{\tilde{\gamma}}\delta^{\gamma}_{\tilde{\beta}}\delta^{\delta}_{\tilde{\alpha}}\Gamma^{\nu]}\psi_\nu \,+\, 12\cdot 120\,\bar{\epsilon}\,\delta^{[\alpha}_{\tilde{\delta}}\delta^{\beta}_{\tilde{\gamma}}\delta^{\gamma}_{\tilde{\beta}}\delta^{\delta}_{\tilde{\alpha}}\Gamma^{\nu]}\psi_\nu \right.\\
&\qquad\qquad \left. +\,96\cdot 6\,\bar{\epsilon}\,\delta^{\alpha}_{\tilde{\alpha}}\delta^{\gamma}_{\tilde{\delta}}\delta^{\delta}_{\tilde{\gamma}}\Gamma_{\tilde{\beta}}\psi^{\beta}\right\} G^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}}\,G_{\alpha\beta\gamma\delta}\\
&= -\frac{9}{2\cdot(12)^3}\left\{ 10\,\bar{\epsilon}\,\delta^{[\alpha}_{\tilde{\delta}}\delta^{\beta}_{\tilde{\gamma}}\delta^{\gamma}_{\tilde{\beta}}\delta^{\delta}_{\tilde{\alpha}}\Gamma^{\nu]}\psi_\nu \,+\, 8\,\bar{\epsilon}\,\delta^{\alpha}_{\tilde{\alpha}}\delta^{\gamma}_{\tilde{\delta}}\delta^{\delta}_{\tilde{\gamma}}\Gamma_{\tilde{\beta}}\psi^{\beta}\right\} G^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}}\,G_{\alpha\beta\gamma\delta}\\
&= -\frac{9}{2\cdot(12)^3}\left\{ -\,8\,(G^2)_{\alpha\beta}\,\bar{\epsilon}\,\Gamma_\alpha\psi^{\beta} \,+\, 10\,\bar{\epsilon}\,\delta^{[\alpha}_{\tilde{\delta}}\delta^{\beta}_{\tilde{\gamma}}\delta^{\gamma}_{\tilde{\beta}}\delta^{\delta}_{\tilde{\alpha}}\Gamma^{\nu]}\psi_\nu\, G^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}}\,G_{\alpha\beta\gamma\delta}\right\}\\
&= -\frac{9}{2\cdot(12)^3}\left\{ -\,16\,(G^2)_{\alpha\beta}\,\bar{\epsilon}\,\Gamma_\alpha\psi^{\beta} \,+\, 2\,(G^2)\,g_{\alpha\beta}\,\bar{\epsilon}\,\Gamma_\alpha\psi^{\beta}\right\}\\
&= \frac{1}{24}\left\{ (G^2)_{\alpha\beta} \,-\, \frac{1}{8}\,(G^2)\,g_{\alpha\beta}\right\}\bar{\epsilon}\,\Gamma_\alpha\psi^{\beta}\ .
\end{aligned}
$$

Collecting the terms in (4.20) with just two (or nine) Gamma matrices and using (2.63) one finds

$$
\begin{aligned}
\bar{\epsilon}\Gamma^{(2)}\psi &= -\,\frac{3}{4\cdot(12)^3}\,(-6)\,\bar{\epsilon}\,\Gamma^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}\nu\alpha\beta\gamma\delta}\,\psi_\nu\,G_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}}\,G_{\alpha\beta\gamma\delta}\\
&= -\,\frac{3\cdot 6}{4\cdot(12)^3}\,\bar{\epsilon}\,\frac{1}{2!}\,\varepsilon^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}\nu\alpha\beta\gamma\delta\mu\rho}\,\Gamma_{\mu\rho}\,\psi_\nu\,G_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}}\,G_{\alpha\beta\gamma\delta}\\
&= -\,\frac{9}{4\cdot(12)^3}\,\varepsilon^{\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4\mu\nu\rho}\,\bar{\epsilon}\,\Gamma_{[\mu\nu}\,\psi_{\rho]}\,G_{\alpha_1\ldots\alpha_4}\,G_{\beta_1\ldots\beta_4}\ .
\end{aligned}
$$

Collecting the terms in (4.20) with just five Gamma matrices, while skipping the two $G_{[4]}$ factors one finds

$$
\begin{aligned}
\bar{\epsilon}\Gamma^{(5)}\psi &= -\,\frac{3}{4\cdot(12)^3}\,\bar{\epsilon}\left\{ -\,6\cdot 120\,\delta^{[\nu}_{[\tilde{\delta}}\,\delta^{\alpha}_{\tilde{\gamma}}\,\Gamma_{\tilde{\alpha}\tilde{\beta}]}{}^{\beta\gamma\delta]}\,\psi_\nu \,+\, 6\cdot 120\,\delta^{[\nu}_{[\tilde{\delta}}\,\delta^{\alpha}_{\tilde{\gamma}}\,\Gamma_{\tilde{\alpha}\tilde{\beta}]}{}^{\beta\gamma\delta]}\,\psi_\nu \right.\\
&\qquad \left. +\,120\,g_{\mu\alpha}\,\delta^{[\mu}_{[\gamma}\,\Gamma^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}]}{}_{\delta]}\,\delta^{\nu}_{\beta}\,\psi_\nu \,+\, 96\,\delta^{\tilde{\alpha}}_{\alpha}\,\Gamma^{\tilde{\beta}\tilde{\gamma}\tilde{\delta}}{}_{\gamma\delta}\,\delta^{\nu}_{\beta}\,\psi_\nu\right\}\\
&= -\,\frac{3}{4\cdot(12)^2}\,\bar{\epsilon}\left\{ 10\,g_{\mu\alpha}\,\delta^{[\mu}_{[\gamma}\,\Gamma^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}]}{}_{\delta]} \,+\, 8\,\delta^{\tilde{\alpha}}_{\alpha}\,\Gamma^{\tilde{\beta}\tilde{\gamma}\tilde{\delta}}{}_{\gamma\delta}\right\}\psi_\beta\\
&= 0\ .
\end{aligned}
$$

Inserting all partial results for (I) and (II) back into (4.14) one obtains:

$$\begin{aligned}
\delta_Q S &= \int dx^D \Bigg[\frac{e}{24} \left((G^2)_{\alpha_1\beta_1} - \frac{1}{8} G^2 \cdot g_{\alpha_1\beta_1} \right) \bar{\epsilon}\, \Gamma^{(\alpha_1} \psi^{\beta_1)} \\
&\qquad - \frac{9}{4 \cdot (12)^4} \varepsilon^{\alpha_1 \ldots \alpha_4 \beta_1 \ldots \beta_4 \mu\nu\rho} G_{\alpha_1 \ldots \alpha_4} G_{\beta_1 \ldots \beta_4} \bar{\epsilon}\, \Gamma_{[\mu\nu} \psi_{\rho]} \\
&\qquad - \frac{e}{24} \left((G^2)_{\alpha_1\beta_1} - \frac{1}{8} G^2 \cdot g_{\alpha_1\beta_1} \right) \bar{\epsilon}\, \Gamma^{(\alpha_1} \psi^{\beta_1)} \Bigg] \\
&= \int dx^D \left[- \frac{9}{4 \cdot (12)^4} \varepsilon^{\alpha_1 \ldots \alpha_4 \beta_1 \ldots \beta_4 \mu\nu\rho} G_{\alpha_1 \ldots \alpha_4} G_{\beta_1 \ldots \beta_4} \bar{\epsilon}\, \Gamma_{[\mu\nu} \psi_{\rho]} \right] \qquad (4.21)
\end{aligned}$$

The variation of the action does not vanish. This must be cured by adding to the action (4.14) a suitable term, whose variation produces the same result but with opposite sign. One makes the following ansatz for a compensating term, called Chern-Simons term,

$$S_{CS} = \int dx^D \frac{1}{4 \cdot (12)^4} \varepsilon^{\alpha_1 \ldots \alpha_4 \beta_1 \ldots \beta_4 \mu\nu\rho} G_{\alpha_1 \ldots \alpha_4} G_{\beta_1 \ldots \beta_4} C_{\mu\nu\rho} \qquad (4.22)$$

and for the super transformation of the 3-form potential

$$\delta_Q C_{\mu\nu\rho} = a\, \bar{\epsilon}\, \Gamma_{[\mu\nu} \psi_{\rho]} \ . \qquad (4.23)$$

Performing the variation of the new term with respect to supersymmetry one obtains:

$$\begin{aligned}
\delta_Q S_{CS} &= \int dx^D \Bigg\{ \frac{2}{4 \cdot (12)^4} \varepsilon^{\alpha_1 \ldots \alpha_4 \beta_1 \ldots \beta_4 \mu\nu\rho} \delta_Q G_{\alpha_1 \ldots \alpha_4} G_{\beta_1 \ldots \beta_4} C_{\mu\nu\rho} \\
&\qquad + \frac{1}{4 \cdot (12)^4} \varepsilon^{\alpha_1 \ldots \alpha_4 \beta_1 \ldots \beta_4 \mu\nu\rho} G_{\alpha_1 \ldots \alpha_4} G_{\beta_1 \ldots \beta_4} \delta_Q C_{\mu\nu\rho} \Bigg\}
\end{aligned}$$

and by inserting (4.23) it reads

$$\begin{aligned}
\delta_Q S_{CS} &= \int dx^D \Bigg\{ \frac{2}{4 \cdot (12)^4} \varepsilon^{\alpha_1 \ldots \alpha_4 \beta_1 \ldots \beta_4 \mu\nu\rho} 4\, \partial_{[\alpha_1} a\, \bar{\epsilon}\, \Gamma_{\alpha_2\alpha_3} \psi_{\alpha_4]} G_{\beta_1 \ldots \beta_4} C_{\mu\nu\rho} \\
&\qquad + \frac{1}{4 \cdot (12)^4} \varepsilon^{\alpha_1 \ldots \alpha_4 \beta_1 \ldots \beta_4 \mu\nu\rho} G_{\alpha_1 \ldots \alpha_4} G_{\beta_1 \ldots \beta_4} a\, \bar{\epsilon}\, \Gamma_{[\mu\nu} \psi_{\rho]} \Bigg\} \ .
\end{aligned}$$

A partial integration can be done and due to

$$
\begin{aligned}
\varepsilon^{\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4\mu\nu\rho}\, 4\, a\, \partial_{[\alpha_1}\bar{\epsilon}\Gamma_{\alpha_2\alpha_3}\psi_{\alpha_4]}\, G_{\beta_1\ldots\beta_4}\, C_{\mu\nu\rho} &= 4\, \partial_{\alpha_1}\left(\varepsilon^{\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4\mu\nu\rho}\, a\, \bar{\epsilon}\Gamma_{\alpha_2\alpha_3}\psi_{\alpha_4}\, G_{\beta_1\ldots\beta_4}\, C_{\mu\nu\rho} \right) \\
&- 4\, \varepsilon^{\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4\mu\nu\rho}\, a\, \bar{\epsilon}\Gamma_{\alpha_2\alpha_3}\psi_{\alpha_4]}\, G_{\beta_1\ldots\beta_4}\, \partial_{[\alpha_1} C_{\mu\nu\rho]} \\
&\sim \varepsilon^{\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4\mu\nu\rho}\, G_{\alpha_1\ldots\alpha_4} G_{\beta_1\ldots\beta_4}\, a\, \bar{\epsilon}\Gamma_{[\mu\nu}\psi_{\rho]}
\end{aligned}
$$

the variation of the Chern-Simons-term reads

$$
\delta_Q \mathcal{L}_{CS} = \frac{3}{4\cdot(12)^4}\, \varepsilon^{\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4\mu\nu\rho}\, G_{\alpha_1\ldots\alpha_4}\, G_{\beta_1\ldots\beta_4}\, a\, \bar{\epsilon}\, \Gamma_{[\mu\nu}\, \psi_{\rho]}\ . \tag{4.24}
$$

This must be compared with the outcome of the variation of the action constructed so far, i.e. with (4.21). Relating the coefficients to each other fixes a to be

$$
a = 3\ . \tag{4.25}
$$

This is consistent with the choice made for the supersymmetric transformation law of the three form (3.2) of section 3 and basically proves it.

The action now reads

$$
\begin{aligned}
\mathcal{L} = &\frac{1}{4}\, eR + \frac{1}{2}\, e\bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu(\omega)\, \psi_\rho - \frac{1}{4\cdot 48}\, eG_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma} \\
&- \frac{1}{4\cdot 48}\, e\left(\bar{\psi}_\mu \Gamma^{\mu\nu\alpha\beta\gamma\delta}\psi_\nu + 12\, \bar{\psi}^\alpha\Gamma^{\gamma\delta}\psi^\beta \right) G_{\alpha\beta\gamma\delta} \\
&+ \frac{1}{4\cdot 144^2}\, \epsilon^{\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4\mu\nu\rho} G_{\alpha_1\ldots\alpha_4} G_{\beta_1\ldots\beta_4} C_{\mu\nu\rho}\ .
\end{aligned} \tag{4.26}
$$

Here we are missing out all four fermion terms, which can be included strangely enough by a simple substitution (c.f. (3.1)).

Finally one has to check once again that the so constructed Lagrangian density with all four fermion terms is invariant under the full action of supersymmetry. The explicit calculation is done in [28] and boils down to checking certain Fierz identities sometimes even more complicated than (2.72).

Homework:

Exercise 4. *Prove (4.17) by transposing (3.8).*

Exercise 5. *Prove (4.18).*

5 Elementary M-Brane Solutions

This chapter is devoted to the presentation of simple solutions to 11d supergravity. We show explicitly that the Einstein equations are satisfied and present some techniques typical of this kind of calculation.

5.1 M5-Brane

The M5-brane solution [30] has the form

$$\begin{aligned} ds^2 &= N^{-1/3}\left(-dt^2 + dx_1^2 + \ldots + dx_5^2\right) \;+\; N^{2/3}\left(dx_6^2 + \ldots + dx_{10}^2\right) \\ G_{\alpha_1\ldots\alpha_4} &= c\, N^{2/3}\, \epsilon_{\alpha_1\ldots\alpha_5}\, \partial^{\alpha_5}\, N(x_6,\ldots,x_{10}), \quad c \;=\; \pm 1, \qquad \Delta N \;=\; 0 \end{aligned} \tag{5.1}$$

and it is convenient to consider the fundamental M5-brane solution as corresponding to the choice $N \;=\; 1 + \frac{a}{r^3}$.

For transparency the metric is written in matrix form:

$$g_{\mu\nu} \;=\; \begin{pmatrix} -N^{-1/3} & & & & & \\ & \ddots & & & & \\ & & N^{-1/3} & & & \\ & & & N^{2/3} & & \\ & & & & \ddots & \\ & & & & & N^{2/3}(x_6\ldots x_{10}) \end{pmatrix} \tag{5.2}$$

5.1.1 Christoffel Symbols

Now we are going to compute the Christoffel symbols of the M5-Brane, defined by

$$\Gamma_{\rho(\nu\lambda)} \;=\; \frac{1}{2}\,\left(g_{\rho\nu,\lambda} + g_{\rho\lambda,\nu} - g_{\nu\lambda,\rho} \right)\;. \tag{5.3}$$

Splitting the index $\mu \in \{0..10\}$ into $a \in \{0..5\}$ and $i \in \{6..10\}$ the Christoffel symbols read:

$$\begin{aligned} \Gamma_{a(bc)} &= 0 \\ \Gamma_{i(bc)} &= -\frac{1}{2}\, g_{bc,i} \;=\; -\,\frac{1}{2}\,\frac{\partial N^{-1/3}}{\partial x^i}\,\eta_{bc} \;=\; \frac{1}{6}\,\frac{\partial_i N}{N^{4/3}}\,\eta_{bc} \\ \Gamma_{b(ic)} &= \frac{1}{2}\, g_{bc,i} \;=\; \frac{1}{2}\,\frac{\partial N^{-1/3}}{\partial x^i}\,\eta_{bc} \;=\; -\,\frac{1}{6}\,\frac{\partial_i N}{N^{4/3}}\,\eta_{bc} \\ \Gamma_{a(ij)} &= \frac{1}{2}\left(g_{ai,j} + g_{aj,i}\right) \;=\; 0 \\ \Gamma_{i(aj)} &= \frac{1}{2}\left(g_{ia,j} - g_{aj,i}\right) \;=\; 0 \\ \Gamma_{i(jk)} &= \frac{1}{2}\left(g_{ij,k} + g_{ik,j} - g_{jk,i}\right) \;=\; \frac{1}{2}\left(\frac{\partial N^{2/3}}{\partial x^k}\,\delta_{ij} + \frac{\partial N^{2/3}}{\partial x^j}\,\delta_{ik} - \frac{\partial N^{2/3}}{\partial x^i}\,\delta_{jk} \right) \\ &= \frac{1}{3}\,\frac{1}{N^{1/3}}\,\left(\partial_k N\,\delta_{ij} + \partial_j N\,\delta_{ik} - \partial_i N\,\delta_{jk} \right) \end{aligned} \tag{5.4}$$

It follows that

$$\Gamma^{\mu}{}_{(\nu\lambda)} \;=\; g^{\mu\rho}\,\Gamma_{\rho(\nu\lambda)} \;=\; g^{\mu a}\Gamma_{a(\nu\lambda)} + g^{\mu i}\Gamma_{i(\nu\lambda)} \tag{5.5}$$

becomes

$$\begin{aligned}
\Gamma^{a}{}_{(bc)} &= g^{ad}\underbrace{\Gamma_{d(bc)}}_{0} + \underbrace{g^{ai}}_{0}\,\Gamma_{i(bc)} \;=\; 0 \\
\Gamma^{i}{}_{(bc)} &= g^{ij}\Gamma_{j(bc)} \;=\; g^{ij}\frac{1}{6}\,\frac{\partial_j N}{N^{4/3}}\,\eta_{bc} \;=\; \frac{1}{6}\frac{\partial^i N}{N^{4/3}}\eta_{bc} \\
\Gamma^{a}{}_{(ib)} &= g^{ac}\Gamma_{c(ib)} \;=\; N^{1/3}\eta^{ac}\left(-\frac{1}{6}\,\frac{\partial_i N}{N^{4/3}}\,\eta_{bc}\right) \;=\; -\frac{1}{6}\frac{\partial_i N}{N}\delta^{a}_{b} \\
\Gamma^{i}{}_{(jb)} &= g^{ik}\Gamma_{k(jb)} \;=\; 0 \\
\Gamma^{a}{}_{(ij)} &= g^{ab}\Gamma_{b(ij)} \;=\; 0 \\
\Gamma^{i}{}_{(jk)} &= g^{il}\Gamma_{l(jk)} \;=\; N^{-2/3}\delta^{il}\frac{1}{3}\,\frac{1}{N^{1/3}}\,\left(\,\partial_k N\,\delta_{lj} + \partial_j N\,\delta_{lk} - \partial_l N\,\delta_{jk}\,\right) \\
&= \frac{1}{3}\,\frac{1}{N}\,\left(\,\partial_k N\,\delta^{i}_{j} + \partial_j N\,\delta^{i}_{k} - \partial^i N\,g_{jk}\,\right)\ .
\end{aligned} \tag{5.6}$$

Of special importance is the partial contraction of the Christoffel symbols, which we add for the convenience of the reader here:

$$\Gamma^{\mu}{}_{(j\mu)} \;=\; \Gamma^{a}{}_{(ja)} + \Gamma^{i}{}_{(ji)} \;=\; -\,\frac{\partial_j N}{N} + \frac{5}{3}\,\frac{\partial_j N}{N} \;=\; \frac{2}{3}\,\frac{\partial_j N}{N}\ . \tag{5.7}$$

5.1.2 Ricci Tensor

The Riemann curvature tensor is defined by

$$R^{\alpha}{}_{\beta\gamma\delta} \;=\; \frac{\partial\Gamma^{\alpha}{}_{(\beta\delta)}}{\partial x^{\gamma}} - \frac{\partial\Gamma^{\alpha}{}_{(\beta\gamma)}}{\partial x^{\delta}} + \Gamma^{\alpha}{}_{(\eta\gamma)}\Gamma^{\eta}{}_{(\beta\delta)} - \Gamma^{\alpha}{}_{(\eta\delta)}\Gamma^{\eta}{}_{(\beta\gamma)}\ . \tag{5.8}$$

A contraction of the first with the third index produces the Ricci tensor:

$$\begin{aligned}
R_{\beta\delta} &= R^{\alpha}{}_{\beta\alpha\delta} \\
&= \frac{\partial\Gamma^{\alpha}{}_{(\beta\delta)}}{\partial x^{\alpha}} - \frac{\partial\Gamma^{\alpha}{}_{(\beta\alpha)}}{\partial x^{\delta}} + \Gamma^{\alpha}{}_{(\eta\alpha)}\Gamma^{\eta}{}_{(\beta\delta)} - \Gamma^{\alpha}{}_{(\eta\delta)}\Gamma^{\eta}{}_{(\beta\alpha)}\ .
\end{aligned} \tag{5.9}$$

According to the split of the indices we compute the components of the Ricci tensor by beginning with a case by case study, now.

(ab)-components:

With eq. (5.7)

$$\begin{aligned}
R_{bd} &= \frac{\partial \Gamma^{i}{}_{(bd)}}{\partial x^{i}} - 0 + \Gamma^{\alpha}{}_{(i\alpha)}\Gamma^{i}{}_{(bd)} - \Gamma^{\alpha}{}_{(\eta d)}\Gamma^{\eta}{}_{(b\alpha)} \\
&= \partial_i \left(\frac{1}{6}\frac{\partial^i N}{N^{4/3}}\eta_{bd} \right) + \frac{2}{3}\,\frac{\partial_i N}{N}\,\frac{1}{6}\frac{\partial^i N}{N^{4/3}}\eta_{bd} - \Gamma^{\alpha}{}_{(\eta d)}\Gamma^{\eta}{}_{(b\alpha)} \\
&= \underbrace{\frac{1}{6}\frac{\partial_i \partial^i N}{N^{4/3}}\eta_{bd}}_{(*)} - \frac{2}{9}\frac{\partial_i N \partial^i N}{N^{7/3}}\eta_{bd} + \frac{2}{3}\,\frac{\partial_i N}{N}\,\frac{1}{6}\frac{\partial^i N}{N^{4/3}}\eta_{bd} - \Gamma^{\alpha}{}_{(\eta d)}\Gamma^{\eta}{}_{(b\alpha)} \qquad (5.10)
\end{aligned}$$

$$\begin{aligned}
(*) &= \frac{1}{6}\frac{\partial_i \partial^i N}{N^{4/3}}\eta_{bd} \;=\; \frac{1}{6}\frac{\partial_i \left(N^{-2/3}\delta^{ij}\,\partial_j N \right)}{N^{4/3}}\eta_{bd} \;=\; \frac{1}{6}\frac{N^{-2/3}\Delta N - 2/3 N^{-5/3}\delta^{ij}\partial_i N \partial_j N}{N^{4/3}}\eta_{bd} \\
&= \frac{1}{6}\frac{0 - 2/3 N^{-1}\partial_i N \partial^i N}{N^{4/3}}\eta_{bd} \;=\; -\frac{1}{9}\frac{\partial_i N \partial^i N}{N^{7/3}}\eta_{bd} \qquad (5.11)
\end{aligned}$$

$$\begin{aligned}
\Gamma^{\alpha}{}_{(\eta d)}\Gamma^{\eta}{}_{(b\alpha)} &= \underbrace{\Gamma^{c}{}_{(ad)}\Gamma^{a}{}_{(bc)}}_{0} + \Gamma^{i}{}_{(ad)}\Gamma^{a}{}_{(bi)} + \Gamma^{c}{}_{(id)}\Gamma^{i}{}_{(bc)} + \underbrace{\Gamma^{j}{}_{(id)}\Gamma^{i}{}_{(bj)}}_{0} \\
&= \left(\frac{1}{6}\frac{\partial^i N}{N^{4/3}}\eta_{ad} \right)\left(-\frac{1}{6}\frac{\partial_i N}{N}\delta^a_b \right) + \left(-\frac{1}{6}\frac{\partial_i N}{N}\delta^c_d \right)\left(\frac{1}{6}\frac{\partial^i N}{N^{4/3}}\eta_{bc} \right) \qquad (5.12)
\end{aligned}$$

$$\begin{aligned}
R_{bd} &= -\frac{1}{9}\frac{\partial_i N \partial^i N}{N^{7/3}}\eta_{bd} - \frac{2}{9}\frac{\partial_i N \partial^i N}{N^{7/3}}\eta_{bd} + \frac{2}{3}\,\frac{\partial_i N}{N}\,\frac{1}{6}\frac{\partial^i N}{N^{4/3}}\eta_{bd} \\
&\quad - \left(\frac{1}{6}\frac{\partial^i N}{N^{4/3}}\eta_{ad} \right)\left(-\frac{1}{6}\frac{\partial_i N}{N}\delta^a_b \right) - \left(-\frac{1}{6}\frac{\partial_i N}{N}\delta^c_d \right)\left(\frac{1}{6}\frac{\partial^i N}{N^{4/3}}\eta_{bc} \right) \\
&= \frac{\partial_i N \partial^i N}{N^{7/3}}\eta_{bd}\left(-\frac{1}{9} - \frac{2}{9} + \frac{1}{9} + \frac{1}{18} \right) \\
&= -\frac{1}{6}\,\frac{\partial_i N \partial^i N}{N^{7/3}}\eta_{bd} \qquad (5.13)
\end{aligned}$$

(ij)-components

$$R_{ij} \;=\; R^{a}{}_{iaj} \,+\, R^{k}{}_{ikj} \qquad (5.14)$$

$$
\begin{aligned}
R^a{}_{iaj} &= \underbrace{\frac{\partial \Gamma^a{}_{(ij)}}{\partial x^a}}_{0} - \frac{\partial \Gamma^a{}_{(ia)}}{\partial x^j} + \Gamma^a{}_{(\eta a)}\Gamma^\eta{}_{(ij)} - \Gamma^a{}_{(\eta j)}\Gamma^\eta{}_{(ia)} \\
&= 0 - \partial_j\left(-\frac{\partial_i N}{N}\right) + \Gamma^a{}_{(ka)}\Gamma^k{}_{(ij)} - \Gamma^a{}_{(bj)}\Gamma^b{}_{(ia)} \\
&= -\partial_j\left(-\frac{\partial_i N}{N}\right) - \frac{\partial_k N}{N}\frac{1}{3}\frac{1}{N}\left(\partial_j N\delta_i^k + \partial_i N\delta_j^k - \partial^k N\, g_{ij}\right) - \left(-\frac{1}{6}\frac{\partial_j N}{N}\delta_b^a\right)\left(-\frac{1}{6}\frac{\partial_i N}{N}\delta_a^b\right) \\
&= \frac{\partial_j\partial_i N}{N} + \frac{\partial_i N\partial_j N}{N^2}\left(-1 - \frac{1}{3} - \frac{1}{3} - \frac{1}{6}\right) + \frac{1}{3}\frac{\partial_k N\partial^k N}{N^2}g_{ij} \qquad (5.15)
\end{aligned}
$$

$$
R^k{}_{ikj} = \underbrace{\frac{\partial \Gamma^k{}_{(ij)}}{\partial x^k}}_{(1)} - \underbrace{\frac{\partial \Gamma^k{}_{(ik)}}{\partial x^j}}_{(2)} + \underbrace{\Gamma^k{}_{(\eta k)}\Gamma^\eta{}_{(ij)}}_{(3)} - \underbrace{\Gamma^k{}_{(\eta j)}\Gamma^\eta{}_{(ik)}}_{(4)} \qquad (5.16)
$$

$$
\begin{aligned}
(1) &= \partial_k\left(\frac{1}{3}\frac{1}{N}\left[\partial_j N\delta_i^k + \partial_i N\delta_j^k - \partial^k N\, g_{ij}\right]\right) \\
&= -\frac{1}{3}\frac{\partial_k N}{N^2}\left[\partial_j N\delta_i^k + \partial_i N\delta_j^k - \partial^k N\, g_{ij}\right] + \frac{1}{3}\frac{1}{N}\left[\partial_k\partial_j N\delta_i^k + \partial_k\partial_i N\delta_j^k - \partial_k\left(\partial^k N\, g_{ij}\right)\right] \\
&= -\frac{1}{3}\frac{1}{N^2}\left[\partial_i N\partial_j N + \partial_i N\partial_j N - \partial_k N\partial^k N\, g_{ij}\right] + \frac{1}{3}\frac{1}{N}\left[\partial_i\partial_j N + \partial_j\partial_i N - \underbrace{\partial_k\left(\partial^k N\, g_{ij}\right)}_{0}\right] \\
&= -\frac{2}{3}\frac{\partial_i N\partial_j N}{N^2} + \frac{1}{3}\frac{\partial_k N\partial^k N}{N^2}g_{ij} + \frac{2}{3}\frac{\partial_i\partial_j N}{N}
\end{aligned}
$$

$$
(2) = \partial_j\left(\frac{5}{3}\frac{\partial_i N}{N}\right) = \frac{5}{3}\frac{\partial_j\partial_i N}{N} - \frac{5}{3}\frac{\partial_i N\partial_j N}{N^2}
$$

$$
\begin{aligned}
(3) &= \frac{5}{3}\frac{\partial_l N}{N}\frac{1}{3}\frac{1}{N}\left(\partial_j N\delta_i^l + \partial_i N\delta_j^l - \partial^l N\, g_{ij}\right) \\
&= \frac{10}{9}\frac{\partial_i N\partial_j N}{N^2} - \frac{5}{9}\frac{\partial_l N\partial^l N}{N^2}g_{ij}
\end{aligned}
$$

$$
\begin{aligned}
(4) &= \frac{1}{3}\frac{1}{N}\left(\partial_j N\delta_l^k + \partial_l N\delta_j^k - \partial^k N\, g_{lj}\right)\cdot\frac{1}{3}\frac{1}{N}\left(\partial_k N\delta_i^l + \partial_i N\delta_k^l - \partial^l N\, g_{ik}\right) \\
&= \frac{1}{9}\frac{1}{N^2}\big(\partial_j N\partial_i N + 5\partial_j N\partial_i N - \partial_j N\partial_i N \\
&\qquad + \partial_i N\partial_j N + \partial_j N\partial_i N - \partial_l N\partial^l N\cdot g_{ij} \\
&\qquad - \partial^k N\partial_k N\cdot g_{ij} - \partial_j N\partial_i N + \partial_i N\partial_j N\big) \\
&= \frac{1}{9}\frac{1}{N^2}\left(7\partial_i N\partial_j N - 2\partial_k N\partial^k N\cdot g_{ij}\right)
\end{aligned}
$$

$$
\begin{aligned}
R^k{}_{ikj} &= -\frac{2}{3}\frac{\partial_i N \partial_j N}{N^2} + \frac{1}{3}\frac{\partial_k N \partial^k N}{N^2}\, g_{ij} + \frac{2}{3}\frac{\partial_i \partial_j N}{N} - \frac{5}{3}\frac{\partial_j \partial_i N}{N} + \frac{5}{3}\frac{\partial_i N \partial_j N}{N^2} \\
&\quad + \frac{10}{9}\frac{\partial_i N \partial_j N}{N^2} - \frac{5}{9}\frac{\partial_l N \partial^l N}{N^2}\, g_{ij} - \frac{1}{9}\frac{1}{N^2}\left(7\partial_i N \partial_j N - 2\partial_k N \partial^k N \cdot g_{ij} \right) \\
&= \frac{\partial_i N \partial_j N}{N^2}\left(-\frac{2}{3} + \frac{5}{3} + \frac{10}{9} - \frac{7}{9} \right) + \frac{\partial_k N \partial^k N}{N^2}\, g_{ij}\left(\frac{1}{3} - \frac{5}{9} + \frac{2}{9} \right) + \frac{\partial_i \partial_j N}{N}\left(\frac{2}{3} - \frac{5}{3} \right) \\
&= \frac{12}{9}\,\frac{\partial_i N \partial_j N}{N^2} - \frac{\partial_i \partial_j N}{N}
\end{aligned}
$$

$$
R_{ij} = -\frac{1}{2}\,\frac{\partial_i N \partial_j N}{N^2} + \frac{1}{3}\frac{\partial_k N \partial^k N}{N^2} g_{ij} \tag{5.17}
$$

The result of the computation can be summarised into the matrix:

$$
R_{\mu\nu} = \begin{pmatrix} -\frac{1}{6}\,\frac{\partial_i N \partial^i N}{N^{7/3}}\eta_{ab} & 0 \\ 0 & -\frac{1}{2}\,\frac{\partial_i N \partial_j N}{N^2} + \frac{1}{3}\frac{\partial_k N \partial^k N}{N^2} g_{ij} \end{pmatrix} \tag{5.18}
$$

5.1.3 Symmetric Field Strength Tensor

We make the following ansatz for the 4-form field strength*7

$$
\begin{aligned}
G_{\alpha_1 \ldots \alpha_4} &= \frac{3}{5}\cdot c \cdot \varepsilon_{\alpha_1 \ldots \alpha_4 \beta}\,\partial^\beta \sqrt{|g_{(5)}|} \;\sim\; *_{(5)}\, d *_{(5)} \mathrm{vol}_{(5)} \\
&= c \cdot N^{2/3}\,\varepsilon_{\alpha_1 \ldots \alpha_4 \beta}\,\partial^\beta N\,, \qquad c = \pm 1
\end{aligned} \tag{5.19}
$$

$$
\begin{aligned}
G_{\alpha_1 \ldots \alpha_4} G^{\alpha_1 \ldots \alpha_4} &= c^2 \cdot N^{4/3}\,\varepsilon_{\alpha_1 \ldots \alpha_4 \beta}\,\varepsilon^{\alpha_1 \ldots \alpha_4 \gamma}\,\partial^\beta N\,\partial_\gamma N \\
&= 4! \cdot c^2 \cdot N^{4/3}\left(\det g_{(5)}\right)^{-1}\partial^\beta N\,\partial_\beta N \\
&= 4! \cdot c^2 \cdot N^{-6/3}\,\partial^\beta N\,\partial_\beta N
\end{aligned} \tag{5.20}
$$

$$
\begin{aligned}
G_{i\alpha_1 \ldots \alpha_3} G_j{}^{\alpha_1 \ldots \alpha_3} &= c^2 \cdot N^{4/3}\,\varepsilon_{i\alpha_1 \ldots \alpha_3 \beta}\, g_{jk} \varepsilon^{k\alpha_1 \ldots \alpha_3 \gamma}\,\partial^\beta N\,\partial_\gamma N \\
&= 3! \cdot c^2 \cdot N^{4/3}\left(\det g_{(5)}\right)^{-1} g_{jk}\,\delta^{k\gamma}_{i\beta}\,\partial^\beta N\,\partial_\gamma N \\
&= 3! \cdot c^2 \cdot N^{-6/3}\, g_{jk}\left(\delta^k_i \delta^\gamma_\beta - \delta^k_\beta \delta^\gamma_i\right)\partial^\beta N\,\partial_\gamma N \\
&= 3! \cdot c^2 \cdot N^{-6/3}\left(g_{ij}\partial^\beta N\,\partial_\beta N - \partial_j N\,\partial_i N \right)
\end{aligned} \tag{5.21}
$$

$$
\frac{1}{12}\left((G^2)_{\mu\nu} - \frac{1}{12}\, g_{\mu\nu}\, G^2 \right) = \begin{pmatrix} -\frac{1}{6}\,\frac{\partial_k N \partial^k N}{N^{6/3}}\, g_{ab} & \\ & -\frac{1}{2}\,\frac{\partial_i N \partial_j N}{N^2} + \frac{1}{3}\,\frac{\partial_k N \partial^k N}{N^2}\, g_{ij} \end{pmatrix}
$$

*7 $\varepsilon_{\alpha_1 \ldots \alpha_4}$ is just the signature of the indices. No factor of $\sqrt{|g|}$!

5.1.4 Killing Spinor Equation

We are now going to investigate the Killing spinor equation (3.10) in the background of an M5-Brane (5.1) and construct the preserved supersymmetries explicitly, i.e. we study solutions of:

$$\hat{D}_M \epsilon \;=\; D_M(\omega)\epsilon - \frac{1}{2\cdot 144}\left(\Gamma^{C_1C_2C_3C_4}{}_M - 8\,\Gamma^{C_2C_3C_4}\delta^{C_1}_M\right)\epsilon\, G_{C_1C_2C_3C_4} \;=\; 0\ . \tag{5.22}$$

with

$$D_M(\omega)\epsilon \;=\; \partial_M \epsilon \,-\, \frac{1}{4}\,\omega_{M\hat{A}\hat{B}}\,\Gamma^{\hat{A}\hat{B}} \tag{5.23}$$

In order to keep control over the different sorts of indices when splitting into longitudinal and transversal directions to the brane we use for eleven dimensional indices capital letters with or without a hat. Hatted indices refer to the tangent frame, while unhatted ones to the spacetime frame. To write out the Killing spinor equation we have to determine the spin connection first. Starting from the definition of the action of the covariant derivative on a vector in tangent frame coordinates

$$\nabla X \;=\; \left(\nabla_M\, X^{\hat{A}}\right)\, dx^M \otimes e_{\hat{A}} \tag{5.24}$$

one obtains

$$\begin{aligned}
\nabla X \;&=\; \left(\partial_M\, X^{\hat{A}} \,+\, \omega_{M\hat{B}}{}^{\hat{A}}\, X^{\hat{B}}\right)\, dx^M \otimes e_{\hat{A}} \\
&=\; \left(\partial_M\, \left(e^{\hat{A}}_Q\, X^Q\right) \,+\, \omega_{M\hat{B}}{}^{\hat{A}}\, e^{\hat{B}}_P\, X^P\right)\, dx^M \otimes \left(\,e^N_{\hat{A}}\,\partial_N\,\right) \\
&=\; \left(\partial_M\, \left(e^{\hat{A}}_Q\, X^Q\right) \,+\, \omega_{M\hat{B}}{}^{\hat{A}}\, e^{\hat{B}}_P\, X^P\right)\, dx^M \otimes \left(\,e^N_{\hat{A}}\,\partial_N\,\right) \\
&=\; e^N_{\hat{A}}\,\left(e^{\hat{A}}_Q\,\partial_M\, X^Q \,+\, \partial_M\, e^{\hat{A}}_Q\, X^Q \,+\, \omega_{M\hat{B}}{}^{\hat{A}}\, e^{\hat{B}}_P\, X^P\right)\, dx^M \otimes \partial_N \\
&=\; \left(\partial_M\, X^N \,+\, \left(e^N_{\hat{A}}\,\partial_M\, e^{\hat{A}}_P \,+\, e^N_{\hat{A}}\, e^{\hat{B}}_P\,\omega_{M\hat{B}}{}^{\hat{A}}\right)\, X^P\right)\, dx^M \otimes \partial_N \\
&=\; \left(\,\partial_M\, X^N \,+\, \Gamma^N{}_{MP}\, X^P\right)\, dx^M \otimes \partial_N\ .
\end{aligned}$$

Comparing the different lines it is straightforward to read off the transformation rule, which expresses the spin connection in terms of the Christoffel connection*8:

$$\omega_{M\hat{B}}{}^{\hat{A}} \;=\; e^{\hat{A}}_N\, e^P_{\hat{B}}\,\Gamma^N{}_{MP} \,-\, e^P_{\hat{B}}\,\partial_M\, e^{\hat{A}}_P\ . \tag{5.25}$$

*8 The same result can also be obtained from the identity $\nabla_\mu\, e^a_\nu \,=\, \partial_\mu\, e^a_\nu \,-\, \Gamma^\rho_{(\mu\nu)} e^a_\rho \,+\, \omega_{\mu b}{}^a\, e^b_\nu \,=\, 0$.

The vielbein of the M5-brane is:

$$e_M^{\hat{A}} \;=\; \begin{pmatrix} N^{-1/6} & & & & & \\ & \ddots & & & & \\ & & N^{-1/6} & & & \\ & & & N^{1/3} & & \\ & & & & \ddots & \\ & & & & & N^{1/3} \end{pmatrix} . \tag{5.26}$$

The name spin connection is somewhat misleading. It is basically just the connection representing the Christoffel connection, when acting on tangent frame indices. The special name is due to its pivotal role it plays in the definition of covariant derivatives on spinorial fields, which cannot be defined in terms of the usual Christoffel symbols [31].
Now we consider the split $M = (a, i)$ and $\hat{M} = (\hat{a}, \hat{\imath})$ and compute all components of the spin connection individually.

<u>First we set $M = a$:</u>

$$\begin{aligned} \omega_{a\hat{B}}{}^{\hat{A}} &= e_N^{\hat{A}}\, e_{\hat{B}}^P\, \Gamma^N{}_{aP} \;-\; 0 \\ &= e_b^{\hat{A}}\, e_{\hat{B}}^P\, \Gamma^b{}_{aP} \;+\; e_i^{\hat{A}}\, e_{\hat{B}}^P\, \Gamma^i{}_{aP} \\ &= e_b^{\hat{A}} \left(e_{\hat{B}}^c\, \Gamma^b{}_{ac} + e_{\hat{B}}^j\, \Gamma^b{}_{aj} \right) \;+\; e_i^{\hat{A}} \left(e_{\hat{B}}^c\, \Gamma^i{}_{ac} + e_{\hat{B}}^j\, \Gamma^i{}_{aj} \right) \end{aligned}$$

Inserting here the Christoffel symbols (5.6) computed before one obtains

$$\omega_{a\hat{B}}{}^{\hat{A}} \;=\; -\frac{1}{6}\, e_a^{\hat{A}}\, e_{\hat{B}}^i\, \frac{\partial_i N}{N} \;+\; \frac{1}{6}\, \eta_{ab}\, e_i^{\hat{A}}\, e_{\hat{B}}^b\, \frac{\partial^i N}{N^{4/3}} \tag{5.27}$$

and by splitting the tangent frame indices one finally obtains the following components of the spin connection:

$$\begin{aligned} \omega_{a\hat{b}}{}^{\hat{a}} &= 0 \\ \omega_{a\hat{\jmath}}{}^{\hat{\imath}} &= 0 \\ \omega_{a\hat{\jmath}}{}^{\hat{a}} &= -\frac{1}{6}\, N^{(-\frac{1}{6}-\frac{1}{3})}\, \delta_a^{\hat{a}}\, \delta_{\hat{\jmath}}^i\, \frac{\partial_i N}{N} \;=\; -\frac{1}{6}\, \frac{\partial_{\hat{\jmath}} N}{N}\, e_a^{\hat{a}} \\ \omega_{a\hat{b}}{}^{\hat{\imath}} &= \frac{1}{6}\, e_{a\hat{b}}\, \frac{\partial^{\hat{\imath}} N}{N} \end{aligned} \tag{5.28}$$

Now we set $M = i$:

$$\begin{aligned}
\omega_{i\hat{B}}{}^{\hat{A}} &= e^{\hat{A}}_N \, e^P_{\hat{B}} \, \Gamma^N{}_{iP} \, - \, e^P_{\hat{B}} \, \partial_i \, e^{\hat{A}}_P \\
&= e^{\hat{A}}_a \, e^P_{\hat{B}} \, \Gamma^a{}_{iP} \, + \, e^{\hat{A}}_j \, e^P_{\hat{B}} \, \Gamma^j{}_{iP} \, - \, e^P_{\hat{B}} \, \partial_i \, e^{\hat{A}}_P \\
&= e^{\hat{A}}_a \, \left(e^b_{\hat{B}} \, \Gamma^a{}_{ib} \, + \, e^k_{\hat{B}} \, \Gamma^a{}_{ik} \right) + \, e^{\hat{A}}_j \, \left(e^b_{\hat{B}} \, \Gamma^j{}_{ib} \, + \, e^k_{\hat{B}} \, \Gamma^j{}_{ik} \right) \, - \, e^P_{\hat{B}} \, \partial_i \, e^{\hat{A}}_P
\end{aligned}$$

Comparing with the Christoffel symbols (5.6) we obtain

$$\omega_{i\hat{B}}{}^{\hat{A}} \;=\; e^{\hat{A}}_a \, e^b_{\hat{B}} \left(-\frac{1}{6} \frac{\partial_i N}{N} \, \delta^a_b \right) + \, e^{\hat{A}}_j \, e^k_{\hat{B}} \, \frac{1}{3} \, \frac{1}{N} \, \left(\partial_i N \, \delta^j_k + \partial_k N \, \delta^j_i - \partial^j N \, g_{ik} \right) \, - \, e^P_{\hat{B}} \, \partial_i \, e^{\hat{A}}_P$$

and after splitting the tangent frame indices according to $\hat{M} = (\hat{a}, \hat{\imath})$

$$\begin{aligned}
\omega_{i\hat{b}}{}^{\hat{a}} &= -\frac{1}{6} \, \frac{\partial_i N}{N} \, \delta^{\hat{a}}_{\hat{b}} \, - \, e^c_{\hat{b}} \, \partial_i \, e^{\hat{a}}_c \; \stackrel{!}{=} \; 0 \\
\omega_{i\hat{l}}{}^{\hat{k}} &= e^{\hat{k}}_j \, e^k_{\hat{l}} \, \frac{1}{3} \, \frac{1}{N} \, \left(\partial_i N \, \delta^j_k + \partial_k N \, \delta^j_i - \partial^j N \, g_{ik} \right) \, - \, e^n_{\hat{l}} \, \partial_i \, e^{\hat{k}}_n \\
&= \frac{1}{3} \, \frac{1}{N} \, \left(\partial_{\hat{l}} N \, e^{\hat{k}}_i - \partial^{\hat{k}} N \, e_{i\hat{l}} \right) \\
\omega_{i\hat{l}}{}^{\hat{a}} &= 0 \\
\omega_{i\hat{a}}{}^{\hat{l}} &= 0
\end{aligned} \tag{5.29}$$

We now write the Killing spinor equation (5.22) according to the split of the index $M = (a, i)$ into two separate equations.

<u>M = a:</u> Keeping just the operator with $M = a$ and skipping ϵ the Killing spinor equation reads

$$\hat{D}_a \;=\; \partial_a \, - \, \frac{1}{4} \, \omega_{a\hat{A}\hat{B}} \, \Gamma^{\hat{A}\hat{B}} - \, \frac{1}{288} \left(\, \Gamma^{i_1 i_2 i_3 i_4}{}_a - 8 \, \Gamma^{i_2 i_3 i_4} \, \underbrace{\delta^{i_1}_a}_{0} \, \right) \, G_{i_1 i_2 i_3 i_4} \tag{5.30}$$

According to (5.1) the four form field strength $G^{(4)}$ just lives in the five dimensional transverse space and we specialised the indices accordingly. Then the last term in the Killing spinor equation does not appear. Using the list of the spin connection components computed above and the ansatz of the four form field strength we finally obtain:

$$\hat{D}_a \;=\; \partial_a \, - \, \frac{1}{24} \, \frac{\partial_{\hat{\jmath}} N}{N} \, \Gamma_a{}^{\hat{\jmath}} + \, \frac{1}{24} \, \frac{\partial_{\hat{\jmath}} N}{N} \, \Gamma^{\hat{\jmath}}{}_a - \, \frac{1}{288} \, \Gamma^{i_1 i_2 i_3 i_4}{}_a \, c \, N^{2/3} \, \varepsilon_{i_1 i_2 i_3 i_4 i_5} \, \partial^{i_5} \, N$$

We must still evaluate the term containing the four form field strength. To this purpose we consider

$$\begin{aligned}
\Gamma^{i_1\ldots i_4}{}_a\,\varepsilon_{i_1i_2i_3i_4i_5} &= \Gamma_a\,\Gamma^{i_1\ldots i_4}\,\varepsilon_{i_1i_2i_3i_4i_5} \\
&= \Gamma_a\,e^{i_1}_{\hat{j}_1}\,e^{i_2}_{\hat{j}_2}\,e^{i_3}_{\hat{j}_3}\,e^{i_4}_{\hat{j}_4}\,\Gamma^{\hat{j}_1\ldots\hat{j}_4}\,\varepsilon_{i_1i_2i_3i_4i_5} \\
&= \Gamma_a\,e^{i_1}_{\hat{j}_1}\,e^{i_2}_{\hat{j}_2}\,e^{i_3}_{\hat{j}_3}\,e^{i_4}_{\hat{j}_4}\,\varepsilon_{i_1i_2i_3i_4i_5}\,\frac{-1}{(11-4)!}\,\varepsilon^{\hat{j}_1\ldots\hat{j}_4\hat{B}_1\ldots\hat{B}_7}\,\Gamma_{\hat{B}_1\ldots\hat{B}_7} \\
&= \Gamma_a\,e^{i_1}_{\hat{j}_1}\,e^{i_2}_{\hat{j}_2}\,e^{i_3}_{\hat{j}_3}\,e^{i_4}_{\hat{j}_4}\,\varepsilon_{i_1i_2i_3i_4i_5}\,\frac{-7}{(11-4)!}\,\varepsilon^{\hat{j}_1\ldots\hat{j}_4\hat{k}\hat{b}_1\ldots\hat{b}_6}\,\Gamma_{\hat{k}\hat{b}_1\ldots\hat{b}_6}
\end{aligned}$$

In the last line we simply observed that the seven general indices (B_i) can be considered as made out of six belonging to the longitudinal directions of the brane (b_i) and one transversal to the brane (k). In addition for the index structure at hand

$$\varepsilon^{\hat{j}_1\ldots\hat{j}_4\hat{k}\hat{b}_1\ldots\hat{b}_6} \;=\; \varepsilon^{\hat{j}_1\ldots\hat{j}_4\hat{k}}\cdot\varepsilon^{\hat{b}_1\ldots\hat{b}_6}$$

and we obtain

$$\begin{aligned}
\Gamma^{i_1\ldots i_4}{}_a\,\varepsilon_{i_1i_2i_3i_4i_5} &= \frac{-7}{(11-4)!}\,\Gamma_a\,e^{i_1}_{\hat{j}_1}\,e^{i_2}_{\hat{j}_2}\,e^{i_3}_{\hat{j}_3}\,e^{i_4}_{\hat{j}_4}\,\varepsilon_{i_1i_2i_3i_4i_5}\,\varepsilon^{\hat{j}_1\ldots\hat{j}_4\hat{k}}\cdot\varepsilon^{\hat{b}_1\ldots\hat{b}_6}\,\Gamma_{\hat{k}\hat{b}_1\ldots\hat{b}_6} \\
&= \frac{7\cdot 6!}{(11-4)!}\,\Gamma_a\,e^{i_1}_{\hat{j}_1}\,e^{i_2}_{\hat{j}_2}\,e^{i_3}_{\hat{j}_3}\,e^{i_4}_{\hat{j}_4}\,\varepsilon_{i_1i_2i_3i_4i_5}\,\varepsilon^{\hat{j}_1\ldots\hat{j}_4\hat{k}}\,e^{k}_{\hat{k}}\,\Gamma_{k012345} \\
&= \frac{7\cdot 6!}{(11-4)!}\,\Gamma_a\;\det(e^{-1}_{(5)})\,\varepsilon_{i_1\ldots i_5}\,\varepsilon^{i_1\ldots i_4k}\,\Gamma_{k012345} \\
&= \frac{7\cdot 6!}{(11-4)!}\,\Gamma_a\;\det(e^{-1}_{(5)})\,\delta^{i_1\ldots i_4k}_{i_1\ldots i_4i_5}\,\Gamma_{k012345} \\
&= \frac{7\cdot 6!\cdot 4!}{(11-4)!}\,\Gamma_a\,N^{-5/3}\;\Gamma_{i_5012345} \\
&= 24\,\Gamma_{ai_5}\,N^{-5/3}\;\Gamma_{012345}
\end{aligned}$$

The appearance of the factor $N^{-5/3}$ and of the gamma matrix Γ_{ai_5} makes it possible to simplify the Killing spinor equation considerably. Plugging this back into the Killing spinor equation one obtains:

$$\hat{D}_a \;=\; \partial_a - \frac{1}{12}\,\frac{\partial_{\hat{j}}N}{N}\,\Gamma_a{}^{\hat{j}}\left(\;1\;+\;c\,\Gamma_{012345}\;\right)$$

Since the solution (5.1) does not depend of the coordinates along the brane it is natural to guess that this might hold for the Killing spinor as well. In this case the partial derivative action on the Killing spinor ϵ vanishes and the above equation becomes purely algebraic, i.e.

$$\left(\;1\;+\;c\,\Gamma_{012345}\;\right)\epsilon\;=\;0\quad\Leftrightarrow\quad\epsilon\;=\;f(x_6,\ldots x_{10})\cdot\left(\;1\;-\;c\,\Gamma_{012345}\;\right)\epsilon_0 \tag{5.31}$$

$\underline{\text{M = i:}}$ The second bit of the Killing spinor equation, i.e. with $M = i$ now reads

$$\begin{aligned}
\hat{D}_i &= \partial_i - \frac{1}{4}\,\omega_{i\hat{A}\hat{B}}\,\Gamma^{\hat{A}\hat{B}} - \frac{1}{288}\left(\Gamma^{i_1 i_2 i_3 i_4}{}_i - 8\,\Gamma^{i_2 i_3 i_4}\delta_i^{i_1} \right) G_{i_1 i_2 i_3 i_4} \\
&= \partial_i + \frac{1}{4}\,\frac{2}{3}\,\frac{\partial_{\hat{l}} N}{N}\,\Gamma_i{}^{\hat{l}} - \frac{1}{288}\left(\underbrace{\Gamma^{i_1 i_2 i_3 i_4}{}_i}_{(*)} - 8\,\underbrace{\Gamma^{i_2 i_3 i_4}\delta_i^{i_1}}_{(**)} \right) c\,N^{2/3}\,\varepsilon_{i_1 i_2 i_3 i_4 i_5}\,\partial^{i_5} N
\end{aligned}$$

To simplify $(*)$ we used

$$\begin{aligned}
\Gamma^{i_1 i_2 i_3 i_4}{}_i\,\varepsilon_{i_1 i_2 i_3 i_4 i_5} &= g_{ik}\,e^{i_1}_{\hat{a}_1}\,e^{i_2}_{\hat{a}_2}\,e^{i_3}_{\hat{a}_3}\,e^{i_4}_{\hat{a}_4}\,e^{k}_{\hat{a}_5}\,\Gamma^{\hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4\hat{a}_5}\,\varepsilon_{i_1 i_2 i_3 i_4 i_5} \\
&= g_{ik}\,e^{i_1}_{\hat{a}_1}\,e^{i_2}_{\hat{a}_2}\,e^{i_3}_{\hat{a}_3}\,e^{i_4}_{\hat{a}_4}\,e^{k}_{\hat{a}_5}\,\frac{-1}{(11-5)!}\,\varepsilon^{\hat{a}_1\ldots\hat{a}_5\hat{B}_1\ldots\hat{B}_6}\,\Gamma_{\hat{B}_1\ldots\hat{B}_6}\,\varepsilon_{i_1 i_2 i_3 i_4 i_5} \\
&= g_{ik}\,\det(e^{-1}_{(5)})\,\frac{-1}{(11-5)!}\,\varepsilon^{i_1 i_2 i_3 i_4 k\hat{B}_1\ldots\hat{B}_6}\,\Gamma_{\hat{B}_1\ldots\hat{B}_6}\,\varepsilon_{i_1 i_2 i_3 i_4 i_5} \\
&= g_{ik}\,\det(e^{-1}_{(5)})\,\varepsilon^{i_1 i_2 i_3 i_4 k}\,\Gamma_{012345}\,\varepsilon_{i_1 i_2 i_3 i_4 i_5} \\
&= g_{ik}\,\det(e^{-1}_{(5)})\,\Gamma_{012345}\,4!\,\delta^k_{i_5} \\
&= 24\,N^{-5/3}\,\Gamma_{012345}\,g_{ii_5}
\end{aligned}$$

and to simplify $(**)$ we used

$$\begin{aligned}
-8\,\Gamma^{i_2 i_3 i_4}\,\varepsilon_{ii_2 i_3 i_4 i_5} &= -8\,e^{i_2}_{\hat{a}_2}\,e^{i_3}_{\hat{a}_3}\,e^{i_4}_{\hat{a}_4}\,\Gamma^{\hat{a}_2\hat{a}_3\hat{a}_4}\,\varepsilon_{ii_2 i_3 i_4 i_5} \\
&= -8\,e^{i_2}_{\hat{a}_2}\,e^{i_3}_{\hat{a}_3}\,e^{i_4}_{\hat{a}_4}\,\frac{1}{(11-3)!}\,\varepsilon^{\hat{a}_2\hat{a}_3\hat{a}_4\hat{B}_1\ldots\hat{B}_8}\,\Gamma_{\hat{B}_1\ldots\hat{B}_8}\,\varepsilon_{ii_2 i_3 i_4 i_5} \\
&= -8\,e^{i_2}_{\hat{a}_2}\,e^{i_3}_{\hat{a}_3}\,e^{i_4}_{\hat{a}_4}\begin{pmatrix} 8 \\ 2 \end{pmatrix}\frac{1}{(11-3)!}\,\varepsilon^{\hat{a}_2\hat{a}_3\hat{a}_4\hat{m}\hat{n}\hat{b}_1\ldots\hat{b}_6}\,\Gamma_{\hat{m}\hat{n}\hat{b}_1\ldots\hat{b}_6}\,\varepsilon_{ii_2 i_3 i_4 i_5} \\
&= -8\,e^{i_2}_{\hat{a}_2}\,e^{i_3}_{\hat{a}_3}\,e^{i_4}_{\hat{a}_4}\,e^{m}_{\hat{m}}\,e^{n}_{\hat{n}}\begin{pmatrix} 8 \\ 2 \end{pmatrix}\frac{1}{(11-3)!}\,\varepsilon^{\hat{a}_2\hat{a}_3\hat{a}_4\hat{m}\hat{n}\hat{b}_1\ldots\hat{b}_6}\,\Gamma_{mn\hat{b}_1\ldots\hat{b}_6}\,\varepsilon_{ii_2 i_3 i_4 i_5} \\
&= -8\,\det(e^{-1}_{(5)})\begin{pmatrix} 8 \\ 2 \end{pmatrix}\frac{1}{(11-3)!}\,\varepsilon^{i_2 i_3 i_4 mn\hat{b}_1\ldots\hat{b}_6}\,\Gamma_{mn\hat{b}_1\ldots\hat{b}_6}\,\varepsilon_{ii_2 i_3 i_4 i_5} \\
&= -8\,\det(e^{-1}_{(5)})\begin{pmatrix} 8 \\ 2 \end{pmatrix}\frac{1}{(11-3)!}\,\varepsilon^{i_2 i_3 i_4 mn\hat{b}_1\ldots\hat{b}_6}\,\Gamma_{mn\hat{b}_1\ldots\hat{b}_6}\,\varepsilon_{ii_2 i_3 i_4 i_5} \\
&= 8\,\det(e^{-1}_{(5)})\begin{pmatrix} 8 \\ 2 \end{pmatrix}\frac{2\cdot 3!}{(11-3)!}\,\varepsilon^{\hat{b}_1\ldots\hat{b}_6}\,\Gamma_{ii_5}\Gamma_{\hat{b}_1\ldots\hat{b}_6} \\
&= -\,8\,N^{-5/3}\begin{pmatrix} 8 \\ 2 \end{pmatrix}\frac{2\cdot 3!\cdot 6!}{(11-3)!}\,\Gamma_{ii_5}\Gamma_{012345} \\
&= -\,48\,N^{-5/3}\,\Gamma_{ii_5}\Gamma_{012345}
\end{aligned}$$

Now we put everything together and obtain the following for of the Killing spinor equation

$$\begin{aligned}\hat{D}_i &= \partial_i + \frac{1}{6}\frac{\partial_{\hat{l}}N}{N}\Gamma_i{}^{\hat{l}} - \frac{c}{12}\Gamma_{012345}\frac{\partial_i N}{N} + \frac{c}{6}\frac{\partial^{i_5}N}{N}\Gamma_{ii_5}\Gamma_{012345} \\ &= \partial_i + \frac{1}{12}\frac{\partial_i N}{N} - \frac{1}{12}\frac{\partial_i N}{N}\,(\,1 + c\,\Gamma_{012345}\,) + \frac{1}{6}\frac{\partial_{\hat{l}}N}{N}\Gamma_i{}^{\hat{l}}\,(\,1 + c\,\Gamma_{012345}\,)\end{aligned}$$

Plugging the ansatz for the Killing spinor from (5.31) into this equation we obtain

$$\left(\partial_i + \frac{1}{12}\frac{\partial_i N}{N}\right) f \;=\; 0 \quad\Leftrightarrow\quad f \;=\; N^{-\frac{1}{12}} \tag{5.32}$$

In fact the computation we have done is more complicated than necessary. A significant simplification can be done by choosing a representation of gamma matrices, properly adjusted to the splitting of the eleven dimensional space into six dimensional and five dimensional spaces corresponding to the longitudinal and transverse directions of the M5-Brane (cf. exercise 7).

5.1.5 BPS-Bound

Spherical coordinates for the 5-sphere:

$$\begin{aligned}
x^5 &= r\cdot\cos\vartheta^1 & 0 &< \vartheta^1 \le \pi \\
x^4 &= r\cdot\sin\vartheta^1\cdot cos\vartheta^2 & 0 &< \vartheta^2 \le \pi \\
x^3 &= r\cdot\sin\vartheta^1\cdot sin\vartheta^2\cdot\cos\vartheta^3 & 0 &< \vartheta^3 \le \pi \\
x^2 &= r\cdot\sin\vartheta^1\cdot sin\vartheta^2\cdot\sin\vartheta^3\cdot\cos\vartheta^4 & 0 &< \vartheta^4 \le 2\pi \\
x^1 &= r\cdot\sin\vartheta^1\cdot sin\vartheta^2\cdot\sin\vartheta^3\cdot\sin\vartheta^4
\end{aligned}$$

$$\begin{aligned}
G &= \frac{1}{4!}\,G_{\alpha_1\ldots\alpha_4}\,dx^{\alpha_1}\wedge dx^{\alpha_2}\wedge dx^{\alpha_3}\wedge dx^{\alpha_4} \\
G &= \frac{1}{4!}\,G_{\alpha_1\ldots\alpha_4}\,\frac{\partial\,(\,x^{\alpha_1}, x^{\alpha_2}, x^{\alpha_3}, x^{\alpha_4}\,)}{\partial\,(\,\vartheta^1, \vartheta^2, \vartheta^3, \vartheta^4\,)}\,d\vartheta^1\wedge d\vartheta^2\wedge d\vartheta^3\wedge d\vartheta^4
\end{aligned}$$

$$Q = \int_{S^4} G \;=\; 8\,ac\,\pi^2$$

Spacelike part of the metric $g_{ij} = \delta_{ij} + h_{ij}$ and computing in the flat background δ_{ij} the asymptotically defined energy [32]

$$\mathcal{E} \;=\; \int\limits_{S^4} \left(\partial^n h_{mn} - \partial_m h^i{}_i \right) \frac{x^m}{r}\, d\Omega_{S^4} \;=\; 8\,\pi^2\, a$$

By directly comparing the results for Q and $\mathcal{E}$ one obtains the equality

$$\mathcal{E} \quad = \quad |\, Q\,| \tag{5.33}$$

characteristic for solutions preserving supersymmetry.

Homework:

Exercise 6. *Recompute the components of the spin connection (5.28) and (5.29) of the M5 brane from the equation*

$$d\, e^{\hat{A}} \;+\; \left(\omega_{M\hat{B}}{}^{\hat{A}}\, dx^M \right) \wedge\, e^{\hat{B}} \;=\; 0\;.$$

Exercise 7. $\mathbb{V}$ *and* $\mathbb{W}$ *vector spaces,* $\mathbb{V}$ *even dimensional. Let* $\Gamma_{D+1} = \Gamma_1 \cdot \ldots \cdot \Gamma_D$ *be the chirality operator of the even dimensional Clifford algebra* $\mathrm{Cliff}(\mathbb{V})$*. Then there is a map* j*, which establishes the isomorphism below:*

$$\begin{aligned} \mathrm{Cliff}(\mathbb{V} \oplus \mathbb{W}, \eta \oplus g) \quad &\xrightarrow{\;\cong\;} \quad \mathrm{Cliff}(\mathbb{V}, \eta) \otimes \mathrm{Cliff}(\mathbb{W}, g) \\ j(v \oplus w) \quad &= \quad v \otimes \mathbb{1} + \Gamma_{D+1} \otimes w \end{aligned}$$

Specialising to the case at hand, one obtains:

$$\begin{aligned} \mathrm{Cliff}(\mathbb{R}^{1,5} \oplus \mathbb{R}^5, \eta \oplus g) \quad &\xrightarrow{\;\cong\;} \quad \mathrm{Cliff}(\mathbb{R}^{1,5}, \eta) \otimes \mathrm{Cliff}(\mathbb{R}^5, g) \\ j(v \oplus w) \quad &= \quad v \otimes \mathbb{1} + \Gamma_{D+1} \otimes w \end{aligned}$$

Using this representation of the 11d Clifford algebra redo the computation of the Killing spinor equation of the M5-Brane (cf. [19]).

Exercise 8. *Compute the charge* Q *and the energy* $\mathcal{E}$*.*

5.2 M2-Brane

The M2-brane solution [33] has the form

$$\begin{aligned} ds^2 &= N^{-2/3}\left(-dt^2+dx_1^2+dx_2^2\right) \;+\; N^{1/3}\left(dx_3^2+\ldots+dx_{10}^2\right) \\ G_{012i} &= c\,\frac{\partial_i N}{N^2}, \quad c \;=\; \pm 1, \qquad \Delta N \;=\; 0 \end{aligned} \tag{5.34}$$

and it is convenient to consider the fundamental M2-brane solution as corresponding to the choice $N \;=\; 1+\frac{a}{r^6}$ with $r^2 = x_3^2+\ldots+x_{10}^2$.

5.2.1 Ricci Tensor

To compute the Ricci-Tensor we employ a slightly different technique compared to the direct one used to obtain the Ricci tensor of the M5-Brane in subsection 5.1 before. It turns out to be much more efficient and can be applied to all M- or D- or p-brane solutions occurring in the various supergravities (not necessarily eleven dimensional). The algorithm is very easy to spell out. First one has to factor out the factor, which makes a part of the metric flat. For convenience we have chosen the upper part to be the flat one, i.e.

$$g_{\mu\nu} \;=\; N^{-2/3}\cdot\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & N\cdot\mathbb{1}_8 \end{pmatrix} . \tag{5.35}$$

In Appendix C we have computed the properties of the Ricci tensor with respect to conformal transformations. We can apply this transformation rule to generate the Ricci tensor of the M2-brane by first computing the Ricci tensor of the metric obtained by neglecting the overall conformal factor in (5.35) and an additional observation stating that the Ricci tensor of a product space build out of a flat space X and a curved space Y is block diagonal as shown below

$$R_{\mu\nu}^{X\times Y} \;=\; \begin{pmatrix} 0 & \\ & R_{ij}^{Y} \end{pmatrix} . \tag{5.36}$$

This in mind we proceed as follows. First we choose Y to be the eight dimensional subspace $\{x^3, \ldots, x^{10}\}$ in eq. (5.35) above, i.e.

$$g_{ij}^{(8)} \;=\; N\cdot\delta_{ij} . \tag{5.37}$$

It can be considered as conformally equivalent to flat space, whose Ricci tensor vanishes. Applying

eq. (C.1) once with $D = d\,(= 8)$ and the conformal factor taken to be N we obtain:

$$
\begin{aligned}
R^Y_{ij} &= 0 + \frac{1}{2}\left\{(1-d)\,\partial_j\left(\frac{\partial_i N}{N}\right) + \partial_i\left(\frac{\partial_j N}{N}\right) - \partial_k\left(\frac{\partial^k N}{N}\right)\delta_{ij}\right\} \\
&\quad + \frac{d-2}{2}\left(\frac{\partial_i N\,\partial_j N}{N^2} - \frac{\partial_k N\,\partial^k N}{N^2}\,\delta_{ij}\right) \\
&= \frac{2-d}{2}\,\frac{\partial_i\partial_j\,N}{N} - \frac{1}{2}\,\frac{\Delta\,N}{N}\,\delta_{ij} + \frac{3d-6}{4}\,\frac{\partial_i N\,\partial_j\,N}{N^2} - \frac{d-4}{4}\,\frac{\delta^{kl}\,\partial_k\,N\,\partial_l\,N}{N^2}\,\delta_{ij}
\end{aligned}
\tag{5.38}
$$

where we have to insert $d = 8$ to obtain the result for the eight dimensional metric. To complete the second step we need the Christoffel symbols of the d-dimensional metric produced so far. They are most easily computed and simply read

$$
\begin{aligned}
\Gamma^i{}_{(jk)} &= \frac{1}{2}\,g^{il}\,(\,g_{lj,k} + g_{lk,j} - g_{jk,l}\,) \\
&= \frac{1}{2}\left(\frac{\partial_j N}{N}\,\delta^i_k + \frac{\partial_k N}{N}\,\delta^i_j - \frac{\partial^i N}{N}\,g_{jk}\right)
\end{aligned}
\tag{5.39}
$$

and for the contraction of the Christoffel symbol one gets:

$$
\Gamma^i{}_{ij} = \frac{d}{2}\,\frac{\partial_j N}{N}\;. \tag{5.40}
$$

Now we perform the second conformal rescaling with the conformal factor taken to be $\Omega = N^{-2/3}$ and $D = 11$. This inserted into eq. (C.1) reads

$$
\begin{aligned}
R^{\mathrm{M2}}_{\beta\delta} &= \tilde{R}^Y_{\beta\delta} + \frac{1}{2}\left\{-10\,\tilde{\nabla}_\delta\left(\frac{\partial_\beta\Omega}{\Omega}\right) + \tilde{\nabla}_\beta\left(\frac{\partial_\delta\Omega}{\Omega}\right) - \tilde{\nabla}_\alpha\left(\frac{\tilde{\partial}^\alpha\Omega}{\Omega}\right)\tilde{g}_{\beta\delta}\right\} \\
&\quad + \frac{9}{4}\left(\frac{\partial_\beta\Omega\,\partial_\delta\Omega}{\Omega^2} - \frac{\partial_\eta\Omega\,\tilde{\partial}^\eta\Omega}{\Omega^2}\,\tilde{g}_{\beta\delta}\right)
\end{aligned}
\tag{5.41}
$$

It is useful to split the indices into those from the 3d space along the M2-brane (in which the unscaled metric is flat), and those 8 directions orthogonal to the M2-brane. The former we call a, $b = 0, 1, 2$, the latter i, $j = 3, \ldots, 10$. The parts of the Ricci tensor are then calculated using the rescaling formula. In the flat directions we get, i.e.

$\underline{\beta = a \;\;\&\;\; \delta = b}$:

$$
R_{ab} = \frac{1}{2}\left\{0 + 0 - \tilde{\nabla}_i\left(\frac{\tilde{\partial}^i\Omega}{\Omega}\right)\eta_{ab}\right\} - \frac{11-2}{4}\,\frac{\partial_i\Omega\,\tilde{\partial}^i\Omega}{\Omega^2}\,\eta_{ab}\,. \tag{5.42}
$$

The first two entries disappear since the function N does not depend on these flat coordinates. Straight forward calculation begins by substituting $\Omega = N^{-2/3}$ and using the correct metric to pull up indices

$$
\begin{aligned}
R_{ab} \;&=\; \frac{1}{3}\,\tilde{\nabla}_i \left(\frac{\tilde{\partial}^i N}{N} \right) \eta_{ab} \;-\; \frac{\partial_i N\, \tilde{\partial}^i N}{N^2}\, \eta_{ab} \\
&=\; \frac{1}{3} \left\{ \partial_i \left(\frac{\tilde{\partial}^i N}{N} \right) \;+\; \Gamma^i{}_{ij}\, \frac{\tilde{\partial}^j N}{N} \right\} \eta_{ab} - \frac{\partial_i N\, \tilde{\partial}^i N}{N^2}\, \eta_{ab}\,.
\end{aligned} \tag{5.43}
$$

Since it is crucial to keep in mind which metric was used to pull up the indices, we write them down again by inserting the correct factor. This factor will contribute towards a chain rule, i.e.

$$
\begin{aligned}
R_{ab} \;&=\; \frac{1}{3} \left\{ \partial_i \left(\frac{N^{-1}\partial_i N}{N} \right) \;+\; 4\, \frac{\partial_j N}{N}\, \frac{N^{-1}\partial_j N}{N} \right\} \eta_{ab} \;-\; \frac{\partial_i N}{N}\, \frac{N^{-1}\partial_i N}{N}\, \eta_{ab} \\
&=\; \frac{1}{3} \left\{ \frac{\Delta N}{N^2} \;-\; 2\, \frac{\partial_i N\, \partial_i N}{N^3} \;+\; 4\, \frac{\partial_i N\, \partial_i N}{N^3} \;-\; 3\, \frac{\partial_i N\, \partial_i N}{N^3} \right\} \eta_{ab} \\
&=\; \frac{1}{3} \left\{ \frac{\Delta N}{N^2} \;-\; \frac{\partial_i N\, \partial_i N}{N^3} \right\} \eta_{ab}\,.
\end{aligned} \tag{5.44}
$$

From the first to the second line, we have used the result for the contraction of the Christoffel symbol (5.40).

Next comes the Ricci tensor for the 8d space with indices i, j starting as is usual from the rescaling formula and substituting $\Omega = N^{-2/3}$ (and rearranging terms), i.e.

<u>$\beta = i \;\;\&\;\; \delta = j$:</u>

$$
\begin{aligned}
R_{ij} \;&=\; \tilde{R}_{ij} \;+\; \frac{11-2}{4} \cdot \frac{4}{9} \left(\frac{\partial_i N\, \partial_j N}{N^2} \;-\; \frac{\partial_k N\, \partial_k N}{N^2}\, \delta_{ij} \right) \\
&\qquad\qquad -\frac{1}{3} \left\{ -\,10\, \tilde{\nabla}_j \left(\frac{\partial_i N}{N} \right) \;+\; \tilde{\nabla}_i \left(\frac{\partial_j N}{N} \right) \;-\; \tilde{\nabla}_k \left(\frac{\tilde{\partial}^k N}{N} \right) \tilde{g}_{ij} \right\}\,.
\end{aligned} \tag{5.45}
$$

The term in curly brackets must be treated with care due to the metrics used to pull up the indices on the differential operators. Hence we explicitly write the result after pulling down the indices of the differentials in the curly brackets also using the symmetry in the first two terms

$$
\{\ldots\} \;=\; -\,\frac{1}{3} \left\{ -\,9\, \partial_j \left(\frac{\partial_i N}{N} \right) \;+\; 9\, \tilde{\Gamma}^k{}_{ij}\, \frac{\partial_k N}{N} \;-\; \partial_k \left(\frac{N^{-1}\partial_k N}{N} \right) \tilde{g}_{ij} \;-\; \Gamma^k{}_{kl}\, \frac{\tilde{\partial}^l N}{N}\, \tilde{g}_{ij} \right\}.
$$

Again we have to make use of (5.40) and we have to pay attention to the third term. The last expression thus becomes

$$\{\ldots\} \;=\; -\frac{1}{3}\Bigg\{ -9\,\frac{\partial_j\partial_i N}{N} + 9\,\frac{\partial_i N\,\partial_j N}{N^2} - \frac{\Delta N}{N}\,\delta_{ij} + 2\,\frac{\partial_k N\,\partial_k N}{N^2}\,\delta_{ij} - 4\,\frac{\partial_k N\,\partial_k N}{N^2}\,\delta_{ij}$$
$$+\,9\cdot\frac{1}{2}\left(\frac{\partial_i N}{N}\delta^k_j + \frac{\partial_j N}{N}\delta^k_i - \frac{\tilde{\partial}^k N}{N}\tilde{g}_{ij}\right)\frac{\partial_k N}{N}\Bigg\}\,. \tag{5.46}$$

The full result for the components of the Ricci tensor in the transverse directions is:

$$\begin{aligned} R_{ij} \;=\;& \frac{\partial_i\partial_j\,N}{N}\,(-3+3) + \frac{\Delta\,N}{N}\,\delta_{ij}\left(-\frac{1}{2}+\frac{1}{3}\right) \\ &+ \frac{\partial_i\,N\partial_j\,N}{N^2}\left(\frac{9}{2}+1-\frac{9}{3}-\frac{9}{3}\right) \\ &+ \frac{\partial_k\,N\,\partial_k\,N}{N^2}\,\delta_{ij}\left(-1-1-\frac{2}{3}+\frac{4}{3}+\frac{3}{2}\right) \\ =\;& -\frac{1}{6}\,\frac{\Delta\,N}{N}\,\delta_{ij} - \frac{1}{2}\,\frac{\partial_i\,N\,\partial_j\,N}{N^2} + \frac{1}{6}\,\frac{\partial_k\,N\,\partial_k\,N}{N^2}\,\delta_{ij} \end{aligned} \tag{5.47}$$

Putting the partial results eq. (5.44) and eq. (5.47) together, the Ricci-Tensor reads

$$R_{\mu\nu} \;=\; \begin{pmatrix} \left(\frac{1}{3}\frac{\Delta N}{N^2} - \frac{1}{3}\frac{\partial_i N\partial_i N}{N^3}\right)\eta_{ab} & 0 \\ 0 & -\frac{1}{2}\frac{\partial_i N\partial_j N}{N^2} + \frac{1}{6}\left(\frac{\partial_k N\partial_k N}{N^2} - \frac{\Delta N}{N}\right)\delta_{ij} \end{pmatrix}. \tag{5.48}$$

5.2.2 Symmetric Field Strength Tensor

$$\begin{aligned} G_{\alpha_1\ldots\alpha_4}G^{\alpha_1\ldots\alpha_4} \;=\;& 4\cdot 3!\cdot G_{012i}\,G^{012i} \;=\; 4\cdot 3!\cdot(-N^2)\cdot c^2\cdot\frac{\partial_i N}{N^2}\,\frac{\partial^i N}{N^2} \\ =\;& -24\cdot c^2\cdot\frac{\partial_i N\,\partial^i N}{N^2} \end{aligned} \tag{5.49}$$

$$G_{\mu\alpha_1\ldots\alpha_3}G_\nu{}^{\alpha_1\ldots\alpha_3} \;=\; \begin{cases} 3!\,c^2\,N^{-8/3}\,\partial_k N\,\partial^k N & \text{if } (\mu,\nu)=(0,0) \\ -\,3!\,c^2\,N^{-8/3}\,\partial_k N\,\partial^k N & \text{if } (\mu,\nu)=(1,1) \\ \text{“} & \text{if } (\mu,\nu)=(2,2) \\ -\,3!\,c^2\,N^{-2}\,\partial_i N\,\partial_j N & \text{if } (\mu,\nu)=(i,j) \end{cases}$$

$$= \;-\,3!\,c^2\,N^{-2}\begin{pmatrix} N^{-2/3}\,\partial_i N\partial^i N\,\eta_{ab} & 0 \\ 0 & \partial_i N\,\partial_j N \end{pmatrix} \tag{5.50}$$

$$\frac{1}{12}\left((G^2)_{\mu\nu} - \frac{1}{12}(G^2)\,g_{\mu\nu}\right) \;=\; -\frac{1}{2}c^2 N^{-2}\begin{pmatrix} \frac{2}{3}N^{-2/3}\,\partial_i N\partial^i N\,\eta_{ab} & 0 \\ 0 & \partial_i N\,\partial_j N - \frac{1}{3}\partial_i N\partial^i N\,g_{ij}\end{pmatrix}$$

Comparison of this tensor and the tensor in eq. (5.48) while taking $\Delta N = 0$ into account, proves the equality.

5.2.3 Killing Spinor Equation

The basic steps of the solution of the Killing spinor equation can be found in various places, e.g. [34]. First of all one has to determine the spin connection from

$$de^{\hat{A}} + \omega_{\hat{B}}{}^{\hat{A}} \wedge e^{\hat{B}} = 0\,. \tag{5.51}$$

Evaluation of this identity leads to the spin connection components below. For instance setting

$\underline{\hat{A} = \hat{a}:}$ $\quad e^{\hat{a}} = N^{-1/3}\,dx^a \quad \Rightarrow$

$$de^{\hat{a}} = -\frac{1}{3}\frac{\partial_i N}{N^{4/3}}\,dx^i \wedge dx^a = \left(-\frac{1}{3}\frac{\partial_i N}{N}\,\delta^{\hat{a}}_{\hat{b}}\,dx^i\right) \wedge e^{\hat{b}} = -\,\omega_{\hat{B}}{}^{\hat{a}} \wedge e^{\hat{B}}\,, \tag{5.52}$$

i.e.

$$\omega_{i\hat{b}}{}^{\hat{a}} = \frac{1}{3}\frac{\partial_i N}{N}\,\delta^{\hat{a}}_{\hat{b}}\,. \tag{5.53}$$

One could has rearranged the terms in (5.52) also this way

$$de^{\hat{a}} = -\frac{1}{3}\frac{\partial_i N}{N^{4/3}}\,dx^i \wedge dx^a = \left(+\frac{1}{3}\frac{\partial_{\hat{i}} N}{N}\,e^{\hat{a}}_c\,dx^c\right) \wedge e^{\hat{i}} = -\,\omega_{\hat{B}}{}^{\hat{a}} \wedge e^{\hat{B}}\,, \tag{5.54}$$

i.e.

$$\omega_{c\hat{i}}{}^{\hat{a}} = -\frac{1}{3}\frac{\partial_{\hat{i}} N}{N}\,e^{\hat{a}}_c\,. \tag{5.55}$$

and similarly

$\underline{\hat{A} = \hat{i}:}$ $\quad e^{\hat{i}} = N^{1/6}\,dx^i \quad \Rightarrow$

$$de^{\hat{i}} = = \frac{1}{6}\frac{\partial_j N}{N^{5/6}}\,dx^j \wedge dx^i = \left(\frac{1}{6}\frac{\partial_j N}{N}\,\delta^{\hat{i}}_{\hat{k}}\,dx^j\right) \wedge e^{\hat{k}} = -\,\omega_{\hat{B}}{}^{\hat{i}} \wedge e^{\hat{B}}\,, \tag{5.56}$$

i.e.

$$\omega_{j\hat{k}}{}^{\hat{i}} \;=\; -\frac{1}{6}\,\frac{\partial_j N}{N}\,\delta^{\hat{i}}_{\hat{k}}\;. \tag{5.57}$$

Using the method discussed in exercise 7 to write the 11d Clifford algebra in terms of longitudinal and transverse directions to the M2-Brane one obtains

$$\Gamma^a \;=\; \gamma^a \otimes \tilde{\gamma}^{(9)} \qquad a \;=\; 0,\dots,2 \tag{5.58}$$

$$\Gamma^i \;=\; \mathbb{1} \otimes \tilde{\gamma}^{i} \qquad i \;=\; 3,\dots,10 \tag{5.59}$$

and $\tilde{\gamma}^{(9)}$ is the eight dimensional chirality operator.

<u>M = a:</u> Only the spin connection (5.55) contributes in that case, i.e.

$$\begin{aligned}
\hat{D}_a \;&=\; \partial_a - \frac{1}{4}\,\omega_{a\hat{A}\hat{B}}\,\Gamma^{\hat{A}\hat{B}} - \frac{1}{288}\left(\;\Gamma^{\alpha_1\alpha_2\alpha_3\alpha_4}{}_a - 8\,\Gamma^{\alpha_2\alpha_3\alpha_4}\delta^{\alpha_1}_a\;\right) G_{\alpha_1\alpha_2\alpha_3\alpha_4}\\
&=\; -\frac{1}{4}\,\omega_{a\hat{i}\hat{b}}\,\Gamma^{\hat{i}\hat{b}} - \frac{1}{288}\left(\;\Gamma^{\alpha_1\alpha_2\alpha_3\alpha_4}{}_a - 8\,\Gamma^{\alpha_2\alpha_3\alpha_4}\delta^{\alpha_1}_a\;\right) G_{\alpha_1\alpha_2\alpha_3\alpha_4}\;.
\end{aligned} \tag{5.60}$$

The other terms simplify as follows:

$$\Gamma^{\alpha_1\alpha_2\alpha_3\alpha_4}{}_a\,G_{\alpha_1\alpha_2\alpha_3\alpha_4} \;=\; 4!\cdot\Gamma^{012i}{}_a G_{012i} \;=\; 0. \tag{5.61}$$

$$\Gamma^{\alpha_2\alpha_3\alpha_4}\,\delta^{\alpha_1}_a\,G_{\alpha_1\alpha_2\alpha_3\alpha_4} \;=\; 3\cdot\Gamma^{\alpha_2\alpha_3 i}G_{a\alpha_2\alpha_3 i} \;=\; 3\cdot\varepsilon_{abc}\gamma^{bc}\tilde{\gamma}^i\cdot c\,\frac{\partial_i N}{N^2} \tag{5.62}$$

Inserting the partial results back into eq. (5.60) one obtains

$$0 \;=\; \frac{1}{6}\,\frac{\partial_{\hat{i}} N}{N} e_{a\hat{b}}\,\tilde{\gamma}^{\hat{i}}\gamma^{\hat{b}}\tilde{\gamma}^{(9)} + \frac{1}{12}\,\varepsilon_{abc}\gamma^{bc}\tilde{\gamma}^i\cdot c\,\frac{\partial_i N}{N^2} \tag{5.63}$$

and using

$$\begin{aligned}
\varepsilon_{abc}\gamma^{bc} \;=\; \varepsilon_{abc}\,e^b_{\hat{b}}\,e^c_{\hat{c}}\gamma^{\hat{b}\hat{c}} \;&=\; \varepsilon_{abc}\,e^b_{\hat{b}}\,e^c_{\hat{c}}\,\varepsilon^{\hat{b}\hat{c}\hat{d}}\,\gamma_{\hat{d}}\\
&=\; \varepsilon_{abc}\,e^b_{\hat{b}}\,e^c_{\hat{c}}\,\varepsilon^{\hat{b}\hat{c}\hat{d}}\,e^d_{\hat{d}}\,\gamma_d\\
&=\; \det(e^{-1}_{(3)})\,\varepsilon_{\hat{b}\hat{c}\hat{d}}\,\varepsilon^{\hat{b}\hat{c}\hat{d}}\,\gamma_d\\
&=\; -2\,N\,\gamma_a
\end{aligned}$$

one finally obtains

$$0 \quad = \quad -\frac{c}{6}\,\frac{\partial_{\hat{i}}N}{N}\,\tilde{\gamma}^{\hat{i}}\gamma_a\left(1\,-\,c\,\tilde{\gamma}^{(9)}\right)\ . \tag{5.64}$$

To solve this equation we simply have to make the ansatz

$$\epsilon \quad = \quad f(x^3,\ldots,x^{10})\cdot\left(1\,+\,c\,\tilde{\gamma}^{(9)}\right)\epsilon_0\ . \tag{5.65}$$

Note that due to the duality symmetry (2.63) $\gamma^{(9)}$ is identical to Γ_{012} and in most research papers $\gamma^{(9)}$ is replaced by the latter one. The evaluation of the second half of the Killing spinor equation follows the same lines as in the case of the M5-Brane and is left as an exercise (exercise 10).

Homework:

Exercise 9. *As mentioned before this method is ideal to compute the Ricci tensor of all the elementary brane solutions for various the supergravities not necessarily eleven dimensional. One simply considers the metric*

$$g_{\mu\nu} \quad = \quad N^{\kappa}\cdot\begin{pmatrix} \eta^{(p)}_{ab} & 0 \\ 0 & N^{\delta}\cdot\delta^{D-p}_{ij}\end{pmatrix}\ . \tag{5.66}$$

Adjusting p, D, κ and δ according to the needs one obtains the Ricci tensor, which must be matched by the suitably chosen field strengths (not considered here).

Exercise 10. *Compute the $M\,=\,i$ part of the Killing spinor equation and determine the scaling function f in $\epsilon\,=\,f(x^3,\ldots x^{10})\,\epsilon_0$.*

6 Intersecting Branes

All the fundamental brane solutions described so far can be used to build more complicated metrics by what is known as intersecting branes. The basic example has already been known for some time [35, 36]. The solution we are studying here was found in [37, 38, 39]. A review focusing on intersecting branes alone is [40].

6.1 Intersecting M5-M5-Brane Solution

$$g_{\mu\nu} = \frac{1}{H_1^{1/3}H_2^{1/3}} \begin{pmatrix} -1 & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & 1 & & & & & & & \\ & & & H_2 & & & & & & \\ & & & & H_2 & & & & & \\ & & & & & H_1 & & & & \\ & & & & & & H_1 & & & \\ & & & & & & & H_1H_2 & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & H_1H_2 \end{pmatrix} \tag{6.1}$$

We split the indices $\mu, \nu \in \{\, x^0, \ldots, x^{10} \,\}$ by using the following notation:

$$\begin{aligned} x, y &\in \{\, x^0, x^1, x^2, x^3 \,\} \\ a, b &\in \{\, x^4, x^5 \,\} \\ \tilde{a}, \tilde{b} &\in \{\, x^6, x^7 \,\} \\ i, j &\in \{\, x^8, x^9, x^{10} \,\} = \perp \end{aligned}$$

By direct computation or applying formula (C.1) as illustrated in subsection 5.2 one obtains the Ricci tensor. With $H_1 = H_1(x^8, x^9, x^{10})$ and $H_2 = H_2(x^8, x^9, x^{10})$ we obtain in non covariant notation*9

$$R_{xy} = \frac{1}{6}\left(\frac{\Delta H_1}{H_1^2 H_2} + \frac{\Delta H_2}{H_1 H_2^2} - \frac{\partial_k H_1 \partial_k H_1}{H_1^3 H_2} - \frac{\partial_k H_2 \partial_k H_2}{H_1 H_2^3}\right)\eta_{xy} \tag{6.2}$$

$$R_{ab} = \left(\frac{1}{6}\left[\frac{\Delta H_1}{H_1^2} - \frac{\partial_k H_1 \partial_k H_1}{H_1^3}\right] - \frac{1}{3}\left[\frac{\Delta H_2}{H_1 H_2} - \frac{\partial_k H_2 \partial_k H_2}{H_1 H_2^2}\right]\right)\delta_{ab} \tag{6.3}$$

$$R_{\tilde{a}\tilde{b}} = \left(\frac{1}{6}\left[\frac{\Delta H_2}{H_2^2} - \frac{\partial_k H_2 \partial_k H_2}{H_2^3}\right] - \frac{1}{3}\left[\frac{\Delta H_1}{H_1 H_2} - \frac{\partial_k H_1 \partial_k H_1}{H_1^2 H_2}\right]\right)\delta_{\tilde{a}\tilde{b}} \tag{6.4}$$

$$\begin{aligned} R_{ij} = & -\frac{1}{2}\frac{\partial_i H_1 \partial_j H_1}{H_1^2} - \frac{1}{2}\frac{\partial_i H_2 \partial_j H_2}{H_2^2} + \frac{1}{3}\left(\frac{\partial_k H_1 \partial_k H_1}{H_1^2} + \frac{\partial_k H_2 \partial_k H_2}{H_2^2}\right. \\ & \left. - \frac{\Delta H_1}{H_1} - \frac{\Delta H_2}{H_2}\right)\delta_{ij} \end{aligned} \tag{6.5}$$

*9 Summation over equal indices is understood. Δ denotes the Euclidean Laplacian, i.e $\Delta = \delta^{kl}\partial_k\partial_l$.

The ansatz for the four form field reads

$$
\begin{aligned}
G_{45ij} &= \varepsilon_{ijk}\partial_k H_2 && (6.6)\\
G_{67ij} &= \varepsilon_{ijk}\partial_k H_1 && (6.7)
\end{aligned}
$$

and we obtain

$$
-\frac{1}{144}\, G^2\, g_{\mu\nu} \;=\; -\frac{1}{6}\left(\frac{\partial_k H_1 \partial_k H_1}{H_1^3 H_2} + \frac{\partial_k H_2 \partial_k H_2}{H_1 H_2^3}\right)(H_1 H_2)^{1/3}\, g_{\mu\nu}\ . \tag{6.8}
$$

The other term $(G^2)_{\mu\nu}$ must be computed by splitting the indices and into longitudinal and transversal ones.

$\underline{\mu = i \;\&\; \nu = j}$: Here $i, j \in \{8, 9, 10\}$ and one obtains

$$
\frac{1}{12}\, G_{i\alpha_1\alpha_2\alpha_3} G_j{}^{\alpha_1\alpha_2\alpha_3} \;=\; \frac{1}{2}\left\{\frac{1}{H_1^2}\left(\partial_k H_1 \partial_k H_1\, \delta_{ij} - \partial_i H_1 \partial_j H_1\right) + \frac{1}{H_2^2}\left(\partial_k H_2 \partial_k H_2\, \delta_{ij} - \partial_i H_2 \partial_j H_2\right)\right\}
$$

and the right hand side of the Einstein equation (3.21) for the indices $(\mu\nu) = (i, j)$ reads

$$
\frac{1}{12}\left((G^2)_{ij} - \frac{1}{12}\, G^2\, g_{ij}\right) \;=\; -\frac{1}{2}\,\frac{\partial_i H_1 \partial_j H_1}{H_1^2} - \frac{1}{2}\,\frac{\partial_i H_2 \partial_j H_2}{H_2^2} + \left[\frac{1}{3}\,\frac{\partial_k H_1 \partial_k H_1}{H_1^2} + \frac{1}{3}\,\frac{\partial_k H_2 \partial_k H_2}{H_2^2}\right]\delta_{ij}
$$

$\underline{\mu = a \;\&\; \nu = b}$: Here $a, b \in \{4, 5\}$

$$
\frac{1}{12}\, G_{a\alpha_1\alpha_2\alpha_3} G_b{}^{\alpha_1\alpha_2\alpha_3} \;=\; \frac{1}{2}\,\frac{\partial_k H_2\, \partial_k H_2}{H_1\, H_2^2}\,\delta_{ab}
$$

and if one repeats the calculation for $\tilde{a}, \tilde{b} \in \{6, 7\}$ one obtains

$$
\frac{1}{12}\, G_{\tilde{a}\alpha_1\alpha_2\alpha_3} G_{\tilde{b}}{}^{\alpha_1\alpha_2\alpha_3} \;=\; \frac{1}{2}\,\frac{\partial_k H_1\, \partial_k H_1}{H_1^2\, H_2}\,\delta_{\tilde{a}\tilde{b}}
$$

Proceeding in the same fashion in computing the other tensor components one obtains finally:

$$
T_{\mu\nu} \;=\; \begin{pmatrix}
-\frac{1}{6}\left(\frac{\partial_k H_1 \partial_k H_1}{H_1^3 H_2} + \frac{\partial_k H_2 \partial_k H_2}{H_1 H_2^3}\right)\eta_{xy} & & & \\
& \left(-\frac{1}{6}\,\frac{\partial_k H_1 \partial_k H_1}{H_1^3} + \frac{1}{3}\,\frac{\partial_k H_2 \partial_k H_2}{H_1 H_2^2}\right)\delta_{ab} & & \\
& & \left(-\frac{1}{6}\,\frac{\partial_k H_2 \partial_k H_2}{H_2^3} + \frac{1}{3}\,\frac{\partial_k H_1 \partial_k H_1}{H_2 H_1^2}\right)\delta_{\tilde{a}\tilde{b}} & \\
& & & -\frac{1}{2}\,\frac{\partial_i H_1 \partial_j H_1}{H_1^2} - \frac{1}{2}\,\frac{\partial_i H_2 \partial_j H_2}{H_2^2} + \left[\frac{1}{3}\,\frac{\partial_k H_1 \partial_k H_1}{H_1^2} + \frac{1}{3}\,\frac{\partial_k H_2 \partial_k H_2}{H_2^2}\right]\delta_{ij}
\end{pmatrix}
$$

7 Exotic solutions

Solutions of supergravity theories which preserve 1/2 of the original supersymmetry typically occur in the description of brane solutions. Using the techniques of intersecting brane solutions one can reduce this number quite generically to any fraction $1/2^n$, with n an integer. Other fractions are less simple to understand and are called exotic. In [41, 42, 43] the authors discovered a new solution of 11 dimensional supergravity, which preserved a fraction of $\nu = 5/8$ supersymmetries. In a following paper [23] it was shown that this solution fits into a whole class of solutions and the questions was raised if further solutions with exotic fractions of preserved supersymmetry can be found within this class. In [43] it was confirmed in a case by case study that this actually is the case.
The class of solutions proposed in [23] are

$$\begin{aligned} ds^2_{11} &= -\,(\,dt \,+\, \omega\,)^2 \;+\; ds^2(\mathbb{R}^{10}) \\ G &= -d\omega \wedge \Omega \end{aligned} \tag{7.1}$$

with Ω defined in terms of complex coordinates $z^a \,=\, x^{2a-1} + i\, x^{2a}$ of $\mathbb{C}^5 \,=\, \mathbb{R}^{10}$,

$$\Omega \;=\; \frac{i}{2}\sum_{a=1}^{5} dz^a \wedge d\bar{z}^a \;, \tag{7.2}$$

and $d\omega \,=\, \alpha \,+\, \bar{\alpha} \,\subset\, \Lambda^{2,0}(\mathbb{C}^5) \oplus \Lambda^{0,2}(\mathbb{C}^5)$.

The Einstein equation (3.21) and the equation of motion of the four form field strength (3.23) are given in section 3 while the Killing spinor equation (3.10) can be written as

$$\partial_\mu \varepsilon \,-\, \frac{1}{4}\omega_{\mu a b}\Gamma^{ab}\,\varepsilon \;+\; \frac{1}{288}\left(\Gamma_\mu \Gamma^{\nu_1\nu_2\nu_3\nu_4} \;-\; 12\,\delta_\mu^{\nu_1}\Gamma^{\nu_2\nu_3\nu_4}\right) G_{\nu_1\nu_2\nu_3\nu_4}\varepsilon \;\;=\;\; 0\;. \tag{7.3}$$

7.1 First Example

The holomorphic two form α we have chosen is

$$\alpha \;=\; \frac{\gamma}{2}\left(\,dz^1 \wedge dz^2 \;+\; dz^3 \wedge dz^4\,\right) \tag{7.4}$$

which leads to

$$\omega \;=\; \frac{\gamma}{2}\left(-x^3dx^1 + x^1dx^3 - x^2dx^4 + x^4dx^2 - y^3dy^1 + y^1dy^3 - y^2dy^4 + y^4dy^2\,\right) \tag{7.5}$$

The equations of motion (3.21) and (3.23) are both satisfied. The Ricci tensor in a tangent frame is diagonal and given by $R_{ab} = \gamma^2 \cdot \mathrm{diag}(2, 1/2, \ldots, 1/2, 0, 0)$ while the contractions of the field strength are:

$$G_{c_1c_2c_3c_4}G^{c_1c_2c_3c_4} = 288\,\gamma^2 \tag{7.6}$$

$$G_{ac_1c_2c_3}G_b{}^{c_1c_2c_3} = \begin{cases} 0 & (a,b) = (0,0) \\ 30\,\gamma^2 & (a,b) = (1,1), \ldots, (8,8) \\ 24\,\gamma^2 & (a,b) = (9,9), (10,10) \\ 0 & a \neq b \end{cases} \tag{7.7}$$

Obviously eq. (3.21) is satisfied. The second term in the equation of motion for the four form field strength (3.23) reduces to

$$\frac{1}{2}G \wedge G = 2\,\gamma^2\,\left(2\,dx^{12345678} + dx^{12345691\!0} + dx^{12347891\!0} + dx^{12567891\!0} + dx^{34567891\!0}\right)$$

and comparing this with $d(*G)$ one finds that both add up to zero.

The Killing spinor equation (7.3) can be written more symbolically as

$$\partial_\mu \varepsilon - \mathbb{M}_\mu \varepsilon = 0\,. \tag{7.8}$$

By just solving this equation for the present ansatz one obtains 12 constant Killing spinors. These 12 solutions exhaust the whole set of solutions. Now we give the argument from which we draw this conclusion.

As a preparation we would like to shed some light onto the terms which we gathered in

$$e_c{}^\mu \mathbb{M}_\mu = \frac{1}{4}\underbrace{\omega_{cab}\Gamma^{ab}}_{(3)} - \frac{1}{288}\left(\Gamma_c\underbrace{(\Gamma^{\nu_1\nu_2\nu_3\nu_4}G_{\nu_1\nu_2\nu_3\nu_4})}_{(1)} - 12\,e_c{}^\mu\underbrace{\Gamma^{\nu_2\nu_3\nu_4}G_{\mu\nu_2\nu_3\nu_4}}_{(2)}\right)\,. \tag{7.9}$$

The components of the field strength $G_{\nu_1\nu_2\nu_3\nu_4}$ are constant and those involving the time direction vanish. Also, gamma matrices with upper spatial indices are identical to the tangent frame gamma matrices. We conclude that the expressions (1) and (2) are constant. The spin connection in (3) turns out to be constant too. So the only source of coordinate dependence comes from contraction with the vielbein in the last term. Since the field strength components which include a time direction vanish, the only remaining source for a coordinate dependence is set to zero.

This example gives a simple necessary condition that each solution must satisfy

$$0 = [\,\partial_\mu, \partial_\nu\,]\,\varepsilon = \partial_\mu(\partial_\nu\varepsilon) - \partial_\nu(\partial_\mu\varepsilon) = \partial_\mu(\,M_\nu\,\varepsilon\,) - \partial_\nu(\,M_\mu\,\varepsilon\,)$$

or transferring the $\mathbb{M}$'s to the tangent frame

$$
\begin{aligned}
0 \;=\; [\,\partial_\mu,\,\partial_\nu\,]\,\varepsilon \;&=\; \partial_\mu(\,e_\nu{}^a\,M_a\,\varepsilon) \,-\, \partial_\nu(\,e_\mu{}^b\,M_b\,\varepsilon) \\
&=\; e_\mu{}^a e_\nu{}^b\,\left(2\,\omega_{[ab]}{}^c\,\mathbb{M}_c \;+\; [\,\mathbb{M}_b,\,\mathbb{M}_a\,]\right)\,\varepsilon\;, \qquad (7.10)
\end{aligned}
$$

where $\mathbb{M}_a \;=\; e_a{}^\mu \mathbb{M}_\mu$ and $\omega_{[\mu\nu]}{}^a \;=\; \partial_{[\mu} e_{\nu]}{}^a$. In deriving eq. (7.10) we used the fact that in the present case all $\mathbb{M}_a$ are constant. Since the only non vanishing components of $\omega_{[\mu\nu]}{}^a$ are

$$
\omega_{[13]}{}^0 = \gamma\;, \qquad \omega_{[24]}{}^0 = -\gamma\;, \qquad \omega_{[57]}{}^0 = \gamma\;, \qquad \omega_{[68]}{}^0 = -\gamma \qquad (7.11)
$$

equation (7.10) evaluated for $(a,b) = (0,1)$ just reduces to

$$
0 \;\;=\;\; [\,\mathbb{M}_1,\,\mathbb{M}_0\,]\,\varepsilon\;. \qquad (7.12)
$$

The kernel is most easily calculated by a computer and in this case it turns out to be just the span of the 12 Killing spinors mentioned above. So we have already found all solutions.

7.2 Second Example

We now take the following ansatz for

$$
\alpha \;\;=\;\; \frac{\gamma}{2}\,\left(dz^1\wedge dz^2 \;+\; dz^1\wedge dz^3 \;+\; dz^2\wedge dz^3\right)\;. \qquad (7.13)
$$

In this case the Ricci tensor in the tangent frame is no longer diagonal. Nevertheless the equations of motion are satisfied both for the metric and the four form field strength.

$$
R_{ab} \;\;=\;\; \begin{pmatrix}
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 1 & 0 & 1/2 & 0 & -1/2 & 0 & 0 & \dots & 0 \\
0 & 0 & 1 & 0 & 1/2 & 0 & -1/2 & 0 & \dots & 0 \\
0 & 1/2 & 0 & 1 & 0 & 1/2 & 0 & 0 & \dots & 0 \\
0 & 0 & 1/2 & 0 & 1 & 0 & 1/2 & 0 & \dots & 0 \\
0 & -1/2 & 0 & 1/2 & 0 & 1 & 0 & 0 & \dots & 0 \\
0 & 0 & -1/2 & 0 & 1/2 & 0 & 1 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0
\end{pmatrix}
$$

One finds 16 constant Killing spinors ε_i generating the 16 dimensional kernels of $\mathbb{M}_1$ to $\mathbb{M}_6$. They are contained in the 20 dimensional kernels of $\mathbb{M}_0$ and $\mathbb{M}_7$ to $\mathbb{M}_{10}$*[10]. The form of the 16 constant Killing spinors can be calculated explicitly and is given for convenience in appendix B. The 20 dimensional kernels of $\mathbb{M}_0$ and $\mathbb{M}_7$ to $\mathbb{M}_{10}$ are all equal and the additional four dimensions are spanned by ϑ, which as a consequence of eq. (7.10) satisfies

$$0 \neq \mathbb{M}_i(\vartheta) \subset \mathrm{span}(\varepsilon_1, \ldots \varepsilon_{16}) , \qquad i = 1, \ldots, 6 .$$

In this situation one can apply the following trick to construct a further Killing spinor [41]. Setting

$$\varepsilon = \vartheta + x^i \, \mathbb{M}_i \, \vartheta$$

the Killing spinor equation (7.8) reads:

$$\begin{aligned} \mu = i = 1, \ldots, 6 \qquad & \partial_i \varepsilon - \mathbb{M}_i \varepsilon = \mathbb{M}_i \vartheta - \mathbb{M}_i \vartheta = 0, \\ \mu = a = 0, 7, \ldots, 10 \qquad & \partial_a \varepsilon - \mathbb{M}_a \varepsilon = - \mathbb{M}_a \left(\vartheta + x^i \, \mathbb{M}_i \, \vartheta \right) = 0 . \end{aligned}$$

Finally we obtain 20 Killing spinors, i.e. $\nu = 5/8$ of the original supersymmetries are preserved.

In the above we have presented two solutions, which preserve two different fractions of supersymmetry. The first example preserves 3/8, the later 5/8. The first example preserves a different exotic fraction of supersymmetry than the example already discovered in [23]. We have also tried different ansätze for α. Their precise form and the amount of supersymmetry they preserve can be found in the table below.

α	ν
$dz^1 \wedge dz^2$	5/8
$dz^1 \wedge dz^2 + dz^1 \wedge dz^3$	5/8
$dz^1 \wedge dz^2 + dz^1 \wedge dz^3 + dz^2 \wedge dz^3$	5/8
$dz^1 \wedge dz^2 + dz^3 \wedge dz^4$	3/8
$dz^1 \wedge dz^2 + dz^3 \wedge dz^4 + dz^1 \wedge dz^5$	3/8
$dz^1 \wedge dz^2 + dz^2 \wedge dz^3 + \ldots + dz^5 \wedge dz^1$	3/8

In particular, we have only found solutions which preserve either 3/8 or 5/8 of the original supersymmetry. It would be interesting to see whether this ansatz could also produce solutions with other numbers of preserved supersymmetries.

*[10] Again the dimensions of the kernels are computed using a computer.

8 IIA Action from M-Theory

The bosonic part of the M-Theory action is (cf. section 3)

$$\begin{aligned}\mathcal{L}_{(11)} \quad = \quad & \frac{1}{4}\sqrt{-g}\,R - \frac{1}{4\cdot 48}\sqrt{-g}\,G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma} \\ & + \frac{1}{4\cdot 144^2}\,\epsilon^{\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4\mu\nu\rho}G_{\alpha_1\ldots\alpha_4}G_{\beta_1\ldots\beta_4}C_{\mu\nu\rho}\end{aligned} \tag{8.1}$$

with the vielbein $e^a{}_\mu$ defined with respect to the mostly plus flat 11d metric.

We want to perform a Kaluza-Klein reduction on a circle S^1 [44, 45], i.e. we consider the 11-th direction to be parametrised by the angular coordinate of the circle $\varphi = x^{11}/R \in [0, 2\pi)$. Here R denotes the radius of the S^1.

This case is of special interest due to the importance of the outcome, i.e. IIA supergravity, in string theory. Therefore the reduction shall be performed as explicit as possible. To that purpose all fields are expanded in Fourier modes

$$T_{\mu\nu\ldots} \quad = \quad \sum_{n=0}^{\infty} T^{(i)}_{\mu\nu\ldots} \cdot e^{in\varphi} \tag{8.2}$$

and a truncation to the lowest order in the Fourier expansion is performed afterwards. From an analytical point of view the Fourier expansion is just an orthogonal decomposition of the infinite dimensional Hilbert space of square integrable functions into a sum of one dimensional subspaces,

$$\mathcal{L}(\mathbb{R}) \quad = \quad \bigoplus_{i=0}^{\infty} \mathcal{L}^{(i)}(\mathbb{R}) \ , \tag{8.3}$$

each of which forms an irreducible representation of the group S^1. From a physical point of view this corresponds to a countable set of discrete excitations with different mass in the 11-direction. From the ten dimensional perspective the 11th direction is invisible as long as one can not excite a reasonable amount of these excitations apart from the lowest lying one. The ten dimensional theory corresponds to the truncation to the massless sector. That this truncation makes sense is the crucial point in the procedure. In the case of an S^1-compactification this is unambiguous because

$$\mathcal{L}^{(i)}(\mathbb{R}) \otimes \mathcal{L}^{(j)}(\mathbb{R}) \quad = \quad \mathcal{L}^{(i+j)}(\mathbb{R}) \ . \tag{8.4}$$

Thus the subspace $\mathcal{L}^{(0)}(\mathbb{R})$ is closed under the tensor product of representations and one can separate the massless sector on the level of the action from the massive one. For more general compactification spaces the truncation must be treated with care due to the complicated way in which products of

higher excitations may contribute to the massless excitations (Clebsch Cordon coefficients) [13]. The first step in the reduction is to write the eleven dimensional metric in terms of a ten dimensional metric, a gauge potential and a scalar field, which resemble the degrees of freedom of the eleven dimensional theory. We make the following choice for the metric, which is still completely general:

$$g^{(11)}_{\mu\nu} \;=\; e^{\frac{4}{3}\phi} \begin{pmatrix} e^{-2\phi}\, g^{st}_{\mu\nu} \;+\; A_\mu\, A_\nu & A_\mu \\ A_\nu & 1 \end{pmatrix} . \tag{8.5}$$

Note the following: In order to perform the reduction of the eleven dimensional Ricci scalar it would be advantageous if we could set $\phi = 0$. Only in this situation the famous Kaluza-Klein formula can be applied, i.e.

$$R_{(11)} \;=\; R_{(10)} \;-\; \frac{1}{4}\,(F^{(2)}_{\mu\nu})^2 \;. \tag{8.6}$$

Here $F^{(2)}$ is the field strength of the potential A_μ, i.e. $F^{(2)}_{\mu\nu} \;=\; 2\,\partial_{[\mu}A_{\nu]}$. The basic steps of the proof are compiled for the convenience of the reader in appendix D and exercise 11 is devoted to this important formula, too. In fact one has enough freedom to shift the problem into a shape, so that (8.6) can be applied. We can cope with the exponential factors due to the behaviour of the Ricci scalar under a Weyl scaling (cf. C.2):

$$\tilde{g}_{\mu\nu} \;=\; e^{2\sigma}\, g_{\mu\nu} \;\Rightarrow\; \tilde{R}_{(\mathrm{d})} \;=\; e^{-2\sigma}\;[\;R_{(\mathrm{d})} \;-\; 2\,(d-1)\,\Delta\sigma \;-\; (d-1)\,(d-2)\,\partial_\mu\sigma\,\partial^\mu\sigma\;]\;. \tag{8.7}$$

A first Weyl scaling is done to get rid of the exponential in front of the metric (8.5). We scale by

$$g^{(11)}_{\mu\nu} \;=\; e^{\frac{4}{3}\phi}\,\bar{g}^{(11)}_{\mu\nu}$$

and the action (8.1) reads

$$\begin{aligned} \mathcal{L}_{(11)} \;=\; & \frac{1}{4}\,e^{\frac{22}{3}\phi}\sqrt{-\bar{g}}\,e^{-\frac{4}{3}\phi}\,\Bigg[\;R\;\underbrace{-\;20\cdot\frac{2}{3}\,\Delta\phi}_{+\;\frac{18}{3}\cdot 20\cdot\frac{2}{3}\,(\partial\phi)^2}\;-\;\left(\frac{2}{3}\right)^2\cdot 10\cdot 9\cdot(\partial\phi)^2\;\Bigg] \\ & -\frac{1}{4\cdot 48}\,e^{\frac{22}{3}\phi}\sqrt{-\bar{g}}\,e^{-\frac{16}{3}\phi}\,G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma} \\ & +\frac{1}{4\cdot 144^2}\,\epsilon^{\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4\mu\nu\rho}G_{\alpha_1\ldots\alpha_4}G_{\beta_1\ldots\beta_4}C_{\mu\nu\rho}\;. \end{aligned}$$

The new metric and its inverse are ($\bar{g}^{(11)}_{\mu\delta}\,\bar{g}^{\delta\nu}_{(11)} \;=\; \delta^\nu_\mu$):

$$\bar{g}^{(11)}_{\mu\delta} \;=\; \begin{pmatrix} \tilde{g}_{\mu\delta} \;+\; A_\mu\,A_\delta & A_\mu \\ A_\delta & 1 \end{pmatrix} \quad \bar{g}^{\delta\nu}_{(11)} \;=\; \begin{pmatrix} \tilde{g}^{\delta\nu} & -\,A^\delta \\ -\,A^\nu & 1\;+\;A_\rho A^\rho \end{pmatrix} .$$

Now we perform the Kaluza-Klein reduction. The most difficult part is to rewrite the terms containing forms due to the split of the metric. Luckily there are only two such terms, a contraction of the four form field strength and a Chern-Simons term. We start with the contraction of the four form field strength now. Since the number of p-forms increases significantly, we adopt the usual conventions and denote the degree by a superscript. For example $A^{(1)} = A_\mu \, dx^\mu$. The Kaluza-Klein reduction of $G_{\alpha_1 \ldots \alpha_4} G^{\alpha_1 \ldots \alpha_4}$ is done in four stages, each of which is characterised by the way the 11-direction is singled out:

$$
\begin{aligned}
G_{\alpha_1 \ldots \alpha_4} G^{\alpha_1 \ldots \alpha_4} &= g^{\alpha_1 \beta_1} \, g^{\alpha_2 \beta_2} \, g^{\alpha_3 \beta_3} \, g^{\alpha_4 \beta_4} \, G_{\alpha_1 \ldots \alpha_4} G_{\beta_1 \ldots \beta_4} \\
\mathit{1.Term}: &= 4 \cdot g^{a_1 b_1} \, g^{a_2 b_2} \, g^{a_3 b_3} \quad \underbrace{g^{11,11}}_{1 + (A^{(1)})^2} \;\; \underbrace{G_{a_1 \ldots a_3 \, 11}}_{H^{(3)}_{a_1 \ldots a_3}} G_{b_1 \ldots b_3 \, 11} \; + \; \ldots \\
&= 4 \cdot (\, 1 \, + \, (A^{(1)})^2 \,) \cdot (H^{(3)})^2 \; + \; \ldots \\
\mathit{2.Term}: &= \mathit{1.Term} \; + \; 2 \cdot \left\{ \underbrace{g^{11 b_1}}_{-A^{b_1}} \, g^{a_2 b_2} \, g^{a_3 b_3} \, g^{a_4 b_4} \, G_{11 \, a_2 \ldots a_4} \underbrace{G_{b_1 \ldots b_4}}_{F^{(4)}_{b_1 \ldots b_4}} \right\} \; + \; \ldots \\
&= \mathit{1.Term} \; + \; 2 \cdot 4 \cdot A^{[b_1} H^{b_2 b_3 b_4]} \, F^{(4)}_{b_1 \ldots b_4} \; + \; \ldots \\
\mathit{3.Term}: &= \mathit{1.} - \mathit{2.Term} \; + \; \underbrace{- \, 12 \, A_a \, A^b \, H_{bcd} \, H^{acd}}_{\text{(see App.D)}} \; + \; \ldots \\
\mathit{4.Term}: &= \mathit{1.} - \mathit{3.Term} \; + \; F^{(4)}_{a_1 \ldots a_4} F^{(4) \, a_1 \ldots a_4} \\
\mathit{finally}: &= 4 \, (A^{(1)})^2 \, (H^{(3)})^2 \; + \; 8 \cdot A^{[b_1} H^{b_2 b_3 b_4]} \, F^{(4)}_{b_1 \ldots b_4} \; + \; F^{(4)}_{a_1 \ldots a_4} F^{(4) \, a_1 \ldots a_4} \\
&\quad - 12 \, A_a \, A^b \, H_{bcd} \, H^{acd} \; + \; 4 \, (H^{(3)})^2 \; .
\end{aligned}
$$

Some short remarks to the numerical factors appearing in the evaluation of the first three terms:

1. There are four possibilities to choose a index pair (α, β) to be $(11, 11)$.

2. The problem is symmetric in the both G's. Therefore we get a first factor of two. Then one observes that the remaining sum can be written compactly as $A^{[b_1} H^{b_2 b_3 b_4]}$ only, if one includes a factor of 4 due to the definition of the antisymmetrisation symbol.

3. The evaluation of the 3. Term is included in the appendix (D).

Finally the first three terms of the last expression can be put into a more compact form. This is due to the fact, that the following identity holds:

$$
\begin{aligned}
(*) \quad &= \quad \Big(F^{(4)}_{a_1\ldots a_4} \,+\, c\cdot A_{[a_1} H_{a_2a_3a_4]} \Big) \Big(F^{(4)\,a_1\ldots a_4} \,+\, c\cdot A^{[a_1} H^{a_2a_3a_4]} \Big) \\
&= \quad F^{(4)}_{a_1\ldots a_4} F^{(4)\,a_1\ldots a_4} \,+\, 2\,c\,F^{(4)}_{a_1\ldots a_4}\, A^{[a_1} H^{a_2\ldots a_4]} \,+\, c^2 \underbrace{A_{[a_1} H_{a_2\ldots a_4]}\, A^{[a_1} H^{a_2\ldots a_4]}}_{\text{(see App.D)}} \\
&= \quad \frac{c^2}{4}\,(A^{(1)})^2\,(H^{(3)})^2 \,+\, 2\,c\,F^{(4)}_{a_1\ldots a_4}\, A^{[a_1} H^{a_2\ldots a_4]} \,+\, F^{(4)}_{a_1\ldots a_4} F^{(4)\,a_1\ldots a_4} \\
&\quad\;\; -12\,\frac{c^2}{4^2}\, A_a\, A^b\, H_{bcd}\, H^{acd}\ .
\end{aligned}
$$

Therefore the expression below simplifies to

$$
G_{\alpha_1\ldots\alpha_4} G^{\alpha_1\ldots\alpha_4} \quad = \quad 4\,(H^{(3)})^2 \,+\, \Big(F^{(4)}_{a_1\ldots a_4} \,+\, 4\cdot A^{(1)}_{[a_1} H^{(3)}_{a_2a_3a_4]} \Big)^2\ .
$$

The Chern-Simons term simplifies in the following way

$$
\begin{aligned}
CS \;&\sim\; \varepsilon^{\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4\mu\nu\rho}\, G_{\alpha_1\ldots\alpha_4}\, G_{\beta_1\ldots\beta_4}\, C_{\mu\nu\rho} \\
&=\; \varepsilon^{11\,a_2\ldots a_4 b_1\ldots b_4 ijk}\, \underbrace{G_{11\,a_2\ldots a_4}}_{-\,H^{(3)}_{a_2\ldots a_4}}\, F^{(4)}_{b_1\ldots b_4}\, A^{(3)}_{ijk} \\
&\qquad +\; \varepsilon^{a_1\alpha_2\ldots\alpha_4\beta_1\ldots\beta_4\mu\nu\rho}\, G_{a_1\alpha_2\ldots\alpha_4}\, G_{\beta_1\ldots\beta_4}\, C_{\mu\nu\rho} \\
&\;\;\vdots \\
&=\; -8\,\varepsilon^{a_1\ldots a_3 b_1\ldots b_4 ijk}\, \underbrace{H^{(3)}_{a_1\ldots a_3}}_{3\,\partial_{[a_1} B^{(2)}_{a_2a_3]}}\, F^{(4)}_{b_1\ldots b_4}\, A^{(3)}_{ijk} \\
&\qquad +3\,\varepsilon^{a_1\ldots a_4 b_1\ldots b_4 ij}\, F^{(4)}_{a_1\ldots a_4}\, F^{(4)}_{b_1\ldots b_4}\, \underbrace{C_{ij\,11}}_{B^{(2)}_{ij}} \\
&=\; +6\,\varepsilon^{a_2a_3 b_1\ldots b_4 a_1 ijk}\, B^{(2)}_{a_2a_3}\, F^{(4)}_{b_1\ldots b_4}\, \underbrace{4\,\partial_{[a_1} A^{(3)}_{ijk]}}_{F^{(4)}_{a_1 ijk}} \\
&\qquad +3\,\varepsilon^{a_1\ldots a_4 b_1\ldots b_4 ij}\, F^{(4)}_{a_1\ldots a_4}\, F^{(4)}_{b_1\ldots b_4}\, B^{(2)}_{ij} \\
&=\; 9\,\varepsilon^{a_1\ldots a_4 b_1\ldots b_4 ij}\, F^{(4)}_{a_1\ldots a_4}\, F^{(4)}_{b_1\ldots b_4}\, B^{(2)}_{ij}\ .
\end{aligned}
$$

Since we had prepared all terms for the reduction step we can now apply the formula (8.6) for the Ricci tensor and add it to the other terms evaluated before to obtain the ten dimensional action:

$$
\begin{aligned}
\mathcal{L}_{(10)} \;&=\; \frac{1}{4}\, e^{\frac{22}{3}\phi} \sqrt{-\tilde{g}}\, e^{-\frac{4}{3}\phi} \left[\tilde{R} \;-\; \frac{1}{4}\, (F^{(2)}_{\mu\nu})^2 \;+\; \left(\frac{2}{3}\right)^2 \cdot 90 \cdot (\partial\phi)^2 \right] \\
&\quad - \frac{1}{4\cdot 48}\, e^{\frac{22}{3}\phi} \sqrt{-\tilde{g}}\, e^{-\frac{16}{3}} \left[4\, (H^{(3)})^2 \;+\; \left(F^{(4)}_{a_1 \ldots a_4} \;+\; 4 \cdot A^{(1)}_{[a_1} H^{(3)}_{a_2 a_3 a_4]} \right)^2 \right] \\
&\quad + \frac{9}{4\cdot 144^2}\, \epsilon^{\alpha_1 \ldots \alpha_4 \beta_1 \ldots \beta_4 \mu\nu} F^{(4)}_{\alpha_1 \ldots \alpha_4} F^{(4)}_{\beta_1 \ldots \beta_4} B^{(2)}_{\mu\nu} \;.
\end{aligned}
$$

Note that the metric is still the rescaled one, i.e. $\tilde{g}_{\mu\nu} \;=\; e^{-2\phi}\, g^{(st)}_{\mu\nu}$. A second Weyl scaling completes the reduction:

$$
\begin{aligned}
\mathcal{L}_{(10)} \;&=\; \frac{1}{4}\, e^{-4\phi} \sqrt{-g_{(st)}} \Bigg[e^{2\phi} \left\{ R \,+\, \underbrace{18\, \Delta\phi}_{36\, (\partial\phi)^2} \,-\, 72\, (\partial\phi)^2 \right\} \;-\; \frac{1}{4}\, e^{4\phi}\, (F^{(2)}_{\mu\nu})^2 \\
&\qquad\qquad + \left(\frac{2}{3}\right)^2 \cdot 90 \cdot e^{2\phi}\, (\partial\phi)^2 \Bigg] \\
&\quad - \frac{1}{4\cdot 48}\, e^{2\phi} \cdot e^{-10\phi} \sqrt{-g_{(st)}} \left[4\, e^{6\phi}\, (H^{(3)})^2 \;+\; e^{8\phi} \left(F^{(4)}_{a_1 \ldots a_4} \;+\; 4 \cdot A^{(1)}_{[a_1} H^{(3)}_{a_2 a_3 a_4]} \right)^2 \right] \\
&\quad + \frac{9}{4\cdot 144^2}\, \epsilon^{\alpha_1 \ldots \alpha_4 \beta_1 \ldots \beta_4 \mu\nu} F^{(4)}_{\alpha_1 \ldots \alpha_4} F^{(4)}_{\beta_1 \ldots \beta_4} B^{(2)}_{\mu\nu} \\
&=\; \frac{1}{4}\, e^{-2\phi} \sqrt{-g_{(st)}} \left[R \,+\, 4 \cdot\, (\partial\phi)^2 \;-\; \frac{1}{2\cdot 3!}\, (H^{(3)})^2 \right] \\
&\quad - \; \frac{1}{4\cdot 4} \sqrt{-g_{(st)}}\, (F^{(2)}_{\mu\nu})^2 \;-\; \frac{1}{4\cdot 48} \sqrt{-g_{(st)}} \left(F^{(4)}_{a_1 \ldots a_4} \;+\; 4 \cdot A^{(1)}_{[a_1} H^{(3)}_{a_2 a_3 a_4]} \right)^2 \\
&\quad + \; \frac{9}{4\cdot 144^2}\, \epsilon^{\alpha_1 \ldots \alpha_4 \beta_1 \ldots \beta_4 \mu\nu} F^{(4)}_{\alpha_1 \ldots \alpha_4} F^{(4)}_{\beta_1 \ldots \beta_4} B^{(2)}_{\mu\nu} \;.
\end{aligned}
$$

This is the action in the string frame. It might be interesting to move from the string to the Einstein frame in a last step. This is done by doing the Weyl scaling

$$
g^{st}_{\mu\nu} \;=\; e^{\frac{\phi}{2}}\, g^{E}_{\mu\nu}
$$

and one obtains

$$
\begin{aligned}
\mathcal{L}_{(10)} \;&=\; \frac{1}{4} \sqrt{-g_{(E)}} \left[R \;-\; \frac{1}{2}\, (\partial\phi)^2 \;-\; \frac{1}{2\cdot 3!}\, e^{-\phi}\, (H^{(3)})^2 \right] \\
&\quad - \; \frac{1}{4\cdot 4}\, e^{\frac{3}{2}\phi} \sqrt{-g_{(E)}}\, (F^{(2)}_{\mu\nu})^2 \;-\; \frac{1}{4\cdot 48}\, e^{\frac{\phi}{2}} \sqrt{-g_{(E)}} \left(F^{(4)}_{a_1 \ldots a_4} \;+\; 4 \cdot A^{(1)}_{[a_1} H^{(3)}_{a_2 a_3 a_4]} \right)^2 \\
&\quad + \; \frac{9}{4\cdot 144^2}\, \epsilon^{\alpha_1 \ldots \alpha_4 \beta_1 \ldots \beta_4 \mu\nu} F^{(4)}_{\alpha_1 \ldots \alpha_4} F^{(4)}_{\beta_1 \ldots \beta_4} B^{(2)}_{\mu\nu} \;.
\end{aligned}
$$

Now we rescale all forms by a factor of two to absorb the terms 1/4 in front of them. This is merely a convention but the one we prefer:

$$
\begin{aligned}
\mathcal{L}_{(10)} \quad = \quad & \frac{1}{4}\sqrt{-g_{(E)}}\left[\ R\ -\ \frac{1}{2}(\partial\phi)^2\ -\ \frac{1}{3}e^{-\phi}(H^{(3)})^2\ \right] \\
& -\ \frac{1}{4}e^{\frac{3}{2}\phi}\sqrt{-g_{(E)}}\,(F^{(2)}_{\mu\nu})^2\ -\ \frac{1}{48}e^{\frac{\phi}{2}}\sqrt{-g_{(E)}}\left(F^{(4)}_{a_1\ldots a_4}\ +\ 8\cdot A^{(1)}_{[a_1}H^{(3)}_{a_2a_3a_4]}\right)^2 \\
& +\ \frac{3}{2\cdot(12)^3}\,\epsilon^{\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4\mu\nu}F^{(4)}_{\alpha_1\ldots\alpha_4}F^{(4)}_{\beta_1\ldots\beta_4}B^{(2)}_{\mu\nu}\ .
\end{aligned}
\tag{8.8}
$$

Homework:

Exercise 11. *Prove formula (8.6) guided by the steps given in appendix D.*

Exercise 12. *Derive the equations of motion of the Lagrangian (8.8) for all gauge fields.*

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A Original Action of Cremmer-Julia-Scherk

Lagrangian:

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4\kappa^2}\, eR - \frac{i}{2}\, e\bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu \left(\frac{\omega + \hat{\omega}}{2}\right) \psi_\rho - \frac{1}{48}\, eF_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma} \\
&\quad + \frac{\kappa}{192}\, e \left(\bar{\psi}_\mu \Gamma^{\mu\nu\alpha\beta\gamma\delta} \psi_\nu + 12\, \bar{\psi}^\alpha \Gamma^{\gamma\delta} \psi^\beta\right) \left(F_{\alpha\beta\gamma\delta} + \hat{F}_{\alpha\beta\gamma\delta} \right) \\
&\quad + \frac{2\kappa}{144^2}\, \epsilon^{\alpha_1 \ldots \alpha_4 \beta_1 \ldots \beta_4 \mu\nu\rho} F_{\alpha_1 \ldots \alpha_4} F_{\beta_1 \ldots \beta_4} A_{\mu\nu\rho}\ .
\end{aligned} \tag{A.1}$$

Supersymmetry:

$$\begin{aligned}
\delta_Q e_\mu{}^a &= -i\, \kappa\, \bar{\epsilon}\, \Gamma^a \psi_\mu \\
\delta_Q A_{\mu\nu\rho} &= \frac{3}{2}\, \bar{\epsilon}\, \Gamma_{[\mu\nu} \psi_{\rho]} \\
\delta_Q \psi_\mu &= \frac{1}{\kappa}\, D_\mu(\hat{\omega})\epsilon + \frac{i}{144} \left(\Gamma^{\alpha\beta\gamma\delta}{}_\mu - 8\, \Gamma^{\beta\gamma\delta} \delta^\alpha_\mu\right) \epsilon\, \hat{F}_{\alpha\beta\gamma\delta} \ = \ \frac{1}{\kappa}\, \hat{D}_\mu \epsilon
\end{aligned} \tag{A.2}$$

The signature of the metric is $\eta_{ab} = (1, -1, \ldots, -1)$ and the Γ-matrices are in a purely imaginary representation of the Clifford algebra

$$\{ \Gamma^a, \Gamma^b \} = 2\, \eta^{ab}\, \mathbb{1}_{32}\ .$$

Abbreviations:

$$\begin{aligned}
D_\nu(\omega)\psi_\mu &= \partial_\nu \psi_\mu + \frac{1}{4}\, \omega_{\nu ab}\, \Gamma^{ab}\, \psi_\mu \\
F_{\mu\nu\rho\sigma} &= 4\, \partial_{[\mu}\, A_{\nu\rho\sigma]} \\
\hat{F}_{\mu\nu\rho\sigma} &= F_{\mu\nu\rho\sigma} - 3\, \kappa\, \bar{\psi}_{[\mu} \Gamma_{\nu\rho} \psi_{\sigma]} \\
K_{\mu ab} &= i\, \frac{\kappa^2}{4} \left[-\bar{\psi}_\alpha \Gamma_{\mu ab}{}^{\alpha\beta} \psi_\beta + 2 \left(\bar{\psi}_\mu \Gamma_b \psi_a - \bar{\psi}_\mu \Gamma_a \psi_b + \bar{\psi}_b \Gamma_\mu \psi_a \right) \right] \quad (\text{contorsion}) \\
\omega_{\mu ab} &= \underbrace{\omega^{(0)}_{\mu ab}}_{\text{Christ}} + K_{\mu ab} \\
\hat{\omega}_{\mu ab} &= \omega_{\mu ab} + i\, \frac{\kappa^2}{4}\, \bar{\psi}_\alpha\, \Gamma_{\mu ab}{}^{\alpha\beta}\, \psi_\beta
\end{aligned}$$

B Explicit Killing spinors

$$\begin{aligned}
\varepsilon &= (\,a_1+a_2+a_3, a_4, a_5, a_6, a_7, a_8, -a_2, -a_4, a_9, a_6, a_2+a_3+a_{10}, a_4, a_{11}, a_{12}, a_5+a_{13}+a_{14},\\
&\quad a_6, a_5-a_9+a_{13}, -a_6, a_{10}, a_4, a_{15}, -a_{12}, a_{14}, a_6, a_1, a_4, a_{13}, -a_6, a_{16}, a_8, -a_3, a_4\,)^T\\
\vartheta &= (\,0, a_{17}, 0, a_{18}, 0, a_{19}, 0, -a_{17}, 0, a_{18}, 0, a_{17}, 0, a_{20}, 0, a_{18}, 0, a_{18},\\
&\quad 0, -a_{17}, 0, a_{20}, 0, -a_{18}, 0, -a_{17}, 0, a_{18}, 0, -a_{19}, 0, -a_{17}\,)^T
\end{aligned}$$

C Behaviour of the Ricci tensor under conformal transformations

$$G_{\mu\nu} \;=\; \Omega\,\tilde{g}_{\mu\nu}$$

$$\Gamma_{\rho(\nu\lambda)} \;=\; \frac{1}{2}\left(G_{\rho\nu,\lambda}+G_{\rho\lambda,\nu}-G_{\nu\lambda,\rho}\right) \;=\; \Omega\,\tilde{\Gamma}_{\rho(\nu\lambda)} + \frac{1}{2}\left(\tilde{g}_{\rho\nu}\,\Omega_{,\lambda}+\tilde{g}_{\rho\lambda}\Omega_{,\nu}-\tilde{g}_{\nu\lambda}\Omega_{,\rho}\right)$$

$$\Gamma^{\rho}{}_{(\nu\lambda)} \;=\; \tilde{\Gamma}^{\rho}{}_{(\nu\lambda)} + \frac{1}{2\Omega}\left(\tilde{\delta}^{\rho}_{\nu}\,\Omega_{,\lambda}+\tilde{\delta}^{\rho}_{\lambda}\Omega_{,\nu}-\tilde{g}^{\rho\kappa}\tilde{g}_{\nu\lambda}\Omega_{,\kappa}\right) \;=\; \tilde{\Gamma}^{\rho}{}_{(\nu\lambda)} + S^{\rho}{}_{(\nu\lambda)}$$

$$\begin{aligned}
R^{\alpha}{}_{\beta\gamma\delta} \;&=\; \frac{\partial(\tilde{\Gamma}+S)^{\alpha}{}_{(\beta\delta)}}{\partial x^{\gamma}} - \frac{\partial(\tilde{\Gamma}+S)^{\alpha}{}_{(\beta\gamma)}}{\partial x^{\delta}} + (\tilde{\Gamma}+S)^{\alpha}{}_{(\eta\gamma)}(\tilde{\Gamma}+S)^{\eta}{}_{(\beta\delta)} - (\tilde{\Gamma}+S)^{\alpha}_{(\eta\delta)}(\tilde{\Gamma}+S)^{\eta}{}_{(\beta\gamma)} \\
&=\; \tilde{R}^{\alpha}{}_{\beta\gamma\delta} + S^{\alpha}{}_{\beta\gamma\delta} + \tilde{\Gamma}^{\alpha}{}_{\eta\gamma}S^{\eta}{}_{\beta\delta} + S^{\alpha}{}_{\eta\gamma}\tilde{\Gamma}^{\eta}{}_{\beta\delta} - \tilde{\Gamma}^{\alpha}{}_{\eta\delta}S^{\eta}{}_{\beta\gamma} - S^{\alpha}{}_{\eta\delta}\tilde{\Gamma}^{\eta}{}_{\beta\gamma}
\end{aligned}$$

$$R_{\beta\delta} \;=\; \tilde{R}_{\beta\delta} + S^{\alpha}{}_{\beta\alpha\delta} + \tilde{\Gamma}^{\alpha}{}_{\eta\alpha}S^{\eta}{}_{\beta\delta} + S^{\alpha}{}_{\eta\alpha}\tilde{\Gamma}^{\eta}{}_{\beta\delta} - \tilde{\Gamma}^{\alpha}{}_{\eta\delta}S^{\eta}{}_{\beta\alpha} - S^{\alpha}{}_{\eta\delta}\tilde{\Gamma}^{\eta}{}_{\beta\alpha}$$

1. Calculation of $S^{\alpha}{}_{\beta\alpha\delta}$

$$\begin{aligned}
S^{\alpha}{}_{\beta\alpha\delta} \;&=\; \frac{\partial S^{\alpha}{}_{(\beta\delta)}}{\partial x^{\alpha}} - \frac{\partial S^{\alpha}{}_{(\beta\alpha)}}{\partial x^{\delta}} + S^{\alpha}{}_{(\eta\alpha)}S^{\eta}{}_{(\beta\delta)} - S^{\alpha}_{(\eta\delta)}S^{\eta}{}_{(\beta\alpha)} \\
&=\; \frac{1}{2}\left\{\partial_{\delta}\left(\frac{\partial_{\beta}\Omega}{\Omega}\right) + \partial_{\beta}\left(\frac{\partial_{\delta}\Omega}{\Omega}\right) - \partial_{\alpha}\left(\frac{\tilde{\partial}^{\alpha}\Omega}{\Omega}\,\tilde{g}_{\beta\delta}\right)\right. \\
&\qquad \left. -\,D\cdot\partial_{\delta}\left(\frac{\partial_{\beta}\Omega}{\Omega}\right) - \partial_{\delta}\left(\frac{\partial_{\beta}\Omega}{\Omega}\right) + \partial_{\delta}\left(\frac{\partial_{\beta}\Omega}{\Omega}\right)\right\} + \frac{D}{4}\left\{2\,\frac{\partial_{\beta}\Omega\partial_{\delta}\Omega}{\Omega^2} - \frac{\partial_{\eta}\Omega\tilde{\partial}^{\eta}\Omega}{\Omega^2}\,\tilde{g}_{\beta\delta}\right\} \\
&\qquad -\frac{1}{4}\left\{3\,\frac{\partial_{\beta}\Omega\partial_{\delta}\Omega}{\Omega^2} + \frac{\partial_{\beta}\Omega\partial_{\delta}\Omega}{\Omega^2} - \frac{\partial_{\eta}\Omega\tilde{\partial}^{\eta}\Omega}{\Omega^2}\,\tilde{g}_{\beta\delta} + D\cdot\frac{\partial_{\beta}\Omega\partial_{\delta}\Omega}{\Omega^2} - \frac{\partial_{\beta}\Omega\partial_{\delta}\Omega}{\Omega^2} - \frac{\partial_{\beta}\Omega\partial_{\delta}\Omega}{\Omega^2} - \frac{\partial_{\eta}\Omega\tilde{\partial}^{\eta}\Omega}{\Omega^2}\,\tilde{g}_{\beta\delta}\right\} \\
&=\; \frac{1}{2}\left\{(1-D)\,\partial_{\delta}\left(\frac{\partial_{\beta}\Omega}{\Omega}\right) + \partial_{\beta}\left(\frac{\partial_{\delta}\Omega}{\Omega}\right) - \partial_{\alpha}\left(\frac{\tilde{\partial}^{\alpha}\Omega}{\Omega}\,\tilde{g}_{\beta\delta}\right)\right\} + \frac{D-2}{4}\left(\frac{\partial_{\beta}\Omega\partial_{\delta}\Omega}{\Omega^2} - \frac{\partial_{\eta}\Omega\tilde{\partial}^{\eta}\Omega}{\Omega^2}\,\tilde{g}_{\beta\delta}\right)
\end{aligned}$$

2. Calculation of cross terms:

$$
\begin{aligned}
\tilde{\Gamma}^{\alpha}{}_{\eta\alpha}\, S^{\eta}{}_{\beta\delta} &= \tilde{\Gamma}^{\alpha}{}_{\eta\alpha}\, \frac{1}{2\Omega} \left(\partial_\beta \Omega\, \delta^{\eta}_{\delta} + \partial_\delta \Omega\, \delta^{\eta}_{\beta} - \tilde{\partial}^{\eta}\Omega\, \tilde{g}_{\beta\delta} \right) \\
&= \frac{1}{2\,\Omega} \left(\tilde{\Gamma}^{\alpha}{}_{\delta\alpha}\, \partial_\beta \Omega + \tilde{\Gamma}^{\alpha}{}_{\beta\alpha}\, \partial_\delta \Omega - \tilde{\Gamma}^{\alpha}{}_{\eta\alpha}\, \tilde{\partial}^{\eta}\Omega\, \tilde{g}_{\beta\delta} \right) \\
S^{\alpha}{}_{\eta\alpha}\, \tilde{\Gamma}^{\eta}{}_{\beta\delta} &= \frac{1}{2\Omega} \left(\partial_\eta \Omega\, \delta^{\alpha}_{\alpha} + \partial_\alpha \Omega\, \delta^{\alpha}_{\eta} - \tilde{\partial}^{\alpha}\Omega\, \tilde{g}_{\alpha\eta} \right) \tilde{\Gamma}^{\eta}{}_{\beta\delta} \\
&= \frac{D}{2\,\Omega}\, \partial_\eta \Omega\, \tilde{\Gamma}^{\eta}{}_{\beta\delta} \\
-\,\tilde{\Gamma}^{\alpha}{}_{\eta\delta}\, S^{\eta}{}_{\beta\alpha} &= -\,\tilde{\Gamma}^{\alpha}{}_{\eta\delta}\, \frac{1}{2\Omega} \left(\partial_\beta \Omega\, \delta^{\eta}_{\alpha} + \partial_\alpha \Omega\, \delta^{\eta}_{\beta} - \tilde{\partial}^{\eta}\Omega\, \tilde{g}_{\alpha\beta} \right) \\
&= \frac{1}{2\Omega} \left(-\,\partial_\beta \Omega\, \tilde{\Gamma}^{\alpha}_{\delta\alpha} - \partial_\alpha \Omega\, \tilde{\Gamma}^{\alpha}_{\beta\delta} + \tilde{\partial}^{\eta}\Omega\, \tilde{\Gamma}^{\alpha}_{\eta\delta}\, \tilde{g}_{\alpha\beta} \right) \\
-\,S^{\alpha}{}_{\eta\delta}\, \tilde{\Gamma}^{\eta}{}_{\beta\alpha} &= -\,\frac{1}{2\Omega} \left(\partial_\eta \Omega\, \delta^{\alpha}_{\delta} + \partial_\delta \Omega\, \delta^{\alpha}_{\eta} - \tilde{\partial}^{\alpha}\Omega\, \tilde{g}_{\eta\delta} \right) \tilde{\Gamma}^{\eta}{}_{\beta\alpha} \\
&= \frac{1}{2\Omega} \left(-\,\partial_\eta \Omega\, \tilde{\Gamma}^{\eta}_{\beta\delta} - \partial_\delta \Omega\, \tilde{\Gamma}^{\alpha}_{\beta\alpha} + \tilde{\partial}^{\alpha}\Omega\, \tilde{\Gamma}^{\eta}_{\beta\alpha}\, \tilde{g}_{\eta\delta} \right)
\end{aligned}
$$

And the sum of all the four terms above is

$$
\tilde{\Gamma}^{\alpha}{}_{\eta\alpha}\, S^{\eta}{}_{\beta\delta} + \ldots - S^{\alpha}{}_{\eta\delta}\, \tilde{\Gamma}^{\eta}{}_{\beta\alpha} = \frac{1}{2\Omega} \left((D-2)\, \partial_\eta \Omega\, \tilde{\Gamma}^{\eta}_{\beta\delta} + \tilde{\partial}^{\eta}\Omega \left[\tilde{\Gamma}^{\alpha}_{(\eta\delta)}\, \tilde{g}_{\alpha\beta} + \tilde{\Gamma}^{\alpha}_{(\eta\beta)}\, \tilde{g}_{\alpha\delta} - \tilde{\Gamma}^{\alpha}_{(\eta\alpha)}\, \tilde{g}_{\beta\delta} \right] \right)
$$

Putting the previous steps together, one obtains the formula

$$
\begin{aligned}
R_{\beta\delta} &= \tilde{R}_{\beta\delta} + \frac{1}{2} \left\{ (1-D)\, \partial_\delta \left(\frac{\partial_\beta \Omega}{\Omega} \right) + \partial_\beta \left(\frac{\partial_\delta \Omega}{\Omega} \right) - \partial_\alpha \left(\frac{\tilde{\partial}^{\alpha}\Omega}{\Omega}\, \tilde{g}_{\beta\delta} \right) \right\} \\
&\quad + \frac{D-2}{4} \left(\frac{\partial_\beta \Omega\, \partial_\delta \Omega}{\Omega^2} - \frac{\partial_\eta \Omega\, \tilde{\partial}^{\eta}\Omega}{\Omega^2}\, \tilde{g}_{\beta\delta} \right) \\
&\quad + \frac{1}{2\Omega} \left((D-2)\, \partial_\eta \Omega\, \tilde{\Gamma}^{\eta}_{(\beta\delta)} + \tilde{\partial}^{\eta}\Omega \left[\tilde{\Gamma}^{\alpha}_{(\eta\delta)}\, \tilde{g}_{\alpha\beta} + \tilde{\Gamma}^{\alpha}_{(\eta\beta)}\, \tilde{g}_{\alpha\delta} - \tilde{\Gamma}^{\alpha}_{(\eta\alpha)}\, \tilde{g}_{\beta\delta} \right] \right)
\end{aligned}
$$

or more covariantly

$$
\begin{aligned}
R_{\beta\delta} &= \tilde{R}_{\beta\delta} + \frac{1}{2} \left\{ (1-D)\, \tilde{\nabla}_\delta \left(\frac{\partial_\beta \Omega}{\Omega} \right) + \tilde{\nabla}_\beta \left(\frac{\partial_\delta \Omega}{\Omega} \right) - \tilde{\nabla}_\alpha \left(\frac{\tilde{\partial}^{\alpha}\Omega}{\Omega} \right) \tilde{g}_{\beta\delta} \right\} \\
&\quad + \frac{D-2}{4} \left(\frac{\partial_\beta \Omega\, \partial_\delta \Omega}{\Omega^2} - \frac{\partial_\eta \Omega\, \tilde{\partial}^{\eta}\Omega}{\Omega^2}\, \tilde{g}_{\beta\delta} \right) \qquad \text{(C.1)}
\end{aligned}
$$

Check:

$$R = \frac{1}{\Omega} \left\{ \tilde{R} - (D-1)\,\tilde{\Delta}(\ln\Omega) - \frac{(D-1)(D-2)}{4}\,\partial_\eta(\ln\Omega)\,\tilde{\partial}^\eta(\ln\Omega) \right\} \tag{C.2}$$

D Some Details of Kaluza-Klein Reductions

Basic Kaluza-Klein Formula:

In order to compute the Ricci tensor one has to compute the connection components first. Here the split of the eleven dimensions into ten plus one must be made explicit. With $M = (\mu, 11)$ and $N = (\nu, 11)$ the metric and its inverse read

$$g^{(11)}_{MN} \;=\; \begin{pmatrix} \tilde{g}_{\mu\nu} \;+\; A_\mu \, A_\nu & A_\mu \\ A_\nu & 1 \end{pmatrix} \qquad g^{MN}_{(11)} \;=\; \begin{pmatrix} \tilde{g}^{\mu\nu} & -\, A^\mu \\ -\, A^\nu & 1 \,+\, A_\rho A^\rho \end{pmatrix} \,.$$

and the 11d Christoffel symbols defined as in eq. (5.3) split into

$$\begin{aligned} \Gamma_{\rho(\mu\nu)} \;&=\; \tilde{\Gamma}_{\rho(\mu\nu)} \;+\; \frac{1}{2} \left(A_\mu \, F_{\nu\rho} \;+\; A_\rho \, B_{\nu\mu} \;+\; A_\nu \, F_{\mu\rho} \right) \\ \Gamma_{11(\mu\nu)} \;&=\; \frac{1}{2} \, B_{\mu\nu} \\ \Gamma_{\mu(11\nu)} \;&=\; -\,\frac{1}{2} \, F_{\mu\nu} \;. \end{aligned}$$

Here we introduced for shortness $B_{\mu\nu} \,=\, \partial_\mu A_\nu + \partial_\nu A_\mu$ and $F_{\mu\nu} \,=\, \partial_\mu A_\nu - \partial_\nu A_\mu$. All other Christoffel symbols vanish. Only the computation of the component $\Gamma_{\rho(\mu\nu)}$ contains a tiny difficulty. One must keep in mind that the metric one has to use in eq. (5.3) is $g_{\mu\nu} \,=\, \tilde{g}_{\mu\nu} \,+\, A_\mu \, A_\nu$. This produces the additional terms.

The Christoffel symbols with upper indices defined in eq. (5.5) read

$$\begin{aligned} \Gamma^{\mu}{}_{(\nu\lambda)} \;&=\; \tilde{\Gamma}^{\mu}{}_{(\nu\lambda)} \;-\; \frac{1}{2} \left(F^{\mu}{}_{\nu} \, A_\lambda + F^{\mu}{}_{\lambda} \, A_\nu \right) \\ \Gamma^{\mu}{}_{(11\lambda)} \;&=\; -\,\frac{1}{2} \, F^{\mu}{}_{\lambda} \\ \Gamma^{\mu}{}_{(11\,11)} \;&=\; 0 \\ \Gamma^{11}{}_{(\nu\lambda)} \;&=\; \frac{1}{2} \, B_{\nu\lambda} \;-\; \frac{1}{2} \, A^\mu \, \left(F_{\nu\mu} \, A_\lambda + F_{\lambda\mu} \, A_\nu \right) \;-\; A^\mu \, \tilde{\Gamma}_{\mu(\nu\lambda)} \\ \Gamma^{11}{}_{(11\mu)} \;&=\; \frac{1}{2} \, A^\nu \, F_{\nu\mu} \\ \Gamma^{11}{}_{(11\,11)} \;&=\; 0 \end{aligned}$$

and for the Ricci tensor defined by eq. (5.9) one obtains

$$
\begin{aligned}
R_{\mu\nu} &= \tilde{R}_{\mu\nu} + \frac{1}{2}\,F_{\mu}{}^{\rho}F_{\rho\nu} + \frac{1}{4}\,A_{\mu}\,A_{\nu}\,F_{\lambda\rho}F^{\lambda\rho} + \frac{1}{2}\,(\,A_{\nu}\,\nabla_{\rho}\,F_{\mu}{}^{\rho} + A_{\mu}\,\nabla_{\rho}\,F_{\nu}{}^{\rho}\,) && \text{(D.1)}\\
R_{11\mu} &= \frac{1}{2}\,\nabla_{\nu}\,F_{\mu}{}^{\nu} + \frac{1}{4}\,A_{\mu}\,F_{\lambda\rho}F^{\lambda\rho} && \text{(D.2)}\\
R_{11\,11} &= \frac{1}{4}\,F_{\lambda\rho}F^{\lambda\rho} && \text{(D.3)}
\end{aligned}
$$

The last two components are obtained without problems. The determination of the first one is slightly more involved and we fill some of the steps in here. According to eq. (5.9) we must compute

$$
R_{ij} = \underbrace{\frac{\partial\Gamma^{k}{}_{(ij)}}{\partial x^{k}}}_{(1)} - \underbrace{\frac{\partial\Gamma^{M}{}_{(iM)}}{\partial x^{j}}}_{(2)} + \underbrace{\Gamma^{M}{}_{(NM)}\Gamma^{N}{}_{(ij)}}_{(3)} - \underbrace{\Gamma^{M}{}_{(Nj)}\Gamma^{N}{}_{(iM)}}_{(4)}\;. \qquad \text{(D.4)}
$$

Now we investigate each of the four terms individually.

$$
(1) = \partial_{k}\left[\,\Gamma^{k}{}_{(ij)}\,\right] = \partial_{k}\left[\tilde{\Gamma}^{k}{}_{(ij)} - \frac{1}{2}\,\left(\,F^{k}{}_{i}\,A_{j} + F^{k}{}_{j}\,A_{i}\,\right)\right] \qquad \text{(D.5)}
$$

$$
\begin{aligned}
(2) &= \partial_{j}\left[\,\Gamma^{N}{}_{(iN)}\,\right] = \partial_{j}\left[\,\Gamma^{11}{}_{(i\,11)} + \Gamma^{k}{}_{(ik)}\,\right]\\
&= \partial_{j}\left[\frac{1}{2}\,A^{l}\,F_{li} - \frac{1}{2}\,A^{k}\,F^{k}{}_{i} + \tilde{\Gamma}^{k}{}_{(ik)}\right] = \partial_{j}\left[\tilde{\Gamma}^{k}{}_{(ik)}\right] && \text{(D.6)}
\end{aligned}
$$

$$
\begin{aligned}
(3) &= \Gamma^{M}{}_{(NM)}\Gamma^{N}{}_{(ij)}\\
&= \Gamma^{11}{}_{(N\,11)}\Gamma^{N}{}_{(ij)} + \Gamma^{k}{}_{(N\,k)}\Gamma^{N}{}_{(ij)}\\
&= \underbrace{\Gamma^{11}{}_{(11\,11)}}_{0}\Gamma^{11}{}_{(ij)} + \Gamma^{11}{}_{(k\,11)}\Gamma^{k}{}_{(ij)} + \underbrace{\Gamma^{k}{}_{(11\,k)}}_{0}\Gamma^{11}{}_{(ij)} + \Gamma^{k}{}_{(l\,k)}\Gamma^{l}{}_{(ij)}\\
&= \frac{1}{2}\,A^{l}\,F_{lk}\left[\tilde{\Gamma}^{k}{}_{(ij)} - \frac{1}{2}\,\left(\,F^{k}{}_{i}\,A_{j} + F^{k}{}_{j}\,A_{i}\,\right)\right]\\
&\quad + \left(-\frac{1}{2}\,F^{k}{}_{l}\,A_{k} + \tilde{\Gamma}^{k}{}_{(lk)}\right)\left[-\frac{1}{2}\,\left(\,F^{l}{}_{i}\,A_{j} + F^{l}{}_{j}\,A_{i}\,\right) + \tilde{\Gamma}^{l}{}_{(ij)}\right] && \text{(D.7)}
\end{aligned}
$$

$$
\begin{aligned}
(4) \quad &= \quad \Gamma^{N}{}_{(Mj)}\,\Gamma^{M}{}_{(iN)} \\
&= \quad \Gamma^{11}{}_{(Mj)}\,\Gamma^{M}{}_{(i\,11)} \,+\, \Gamma^{k}{}_{(Mj)}\,\Gamma^{M}{}_{(ik)} \\
&= \quad \Gamma^{11}{}_{(11\,j)}\,\Gamma^{11}{}_{(i\,11)} \,+\, \Gamma^{11}{}_{(lj)}\,\Gamma^{l}{}_{(i\,11)} \,+\, \Gamma^{k}{}_{(11j)}\,\Gamma^{11}{}_{(ik)} \,+\, \Gamma^{k}{}_{(lj)}\,\Gamma^{l}{}_{(ik)} \\
&= \quad \frac{1}{2}\,A^{l}\,F_{lj}\,\frac{1}{2}\,A^{k}\,F_{ki} \,+\, \left[\frac{1}{2}\,B_{kj} \,-\, \frac{1}{2}\,A^{l}\,\left(\,F_{kl}\,A_{j} \,+\, F_{jl}\,A_{k}\,\right) \,-\, A^{l}\,\tilde{\Gamma}_{l(kj)}\right]\,\left[-\frac{1}{2}\,F^{k}{}_{i}\right] \\
&\qquad -\,\frac{1}{2}\,F^{k}{}_{j}\,\left[\frac{1}{2}\,B_{ik} \,-\, \frac{1}{2}\,A^{l}\,\left(\,F_{il}\,A_{k} \,+\, F_{kl}\,A_{i}\,\right) \,-\, A^{l}\,\tilde{\Gamma}_{l(ik)}\right] \\
&\qquad +\,\left[\tilde{\Gamma}^{k}{}_{(lj)} \,-\, \frac{1}{2}\,\left(F^{k}{}_{l}\,A_{j} \,+\, F^{k}{}_{j}\,A_{l}\right)\right]\left[\tilde{\Gamma}^{l}{}_{(ik)} \,-\, \frac{1}{2}\,\left(F^{l}{}_{i}\,A_{k} \,+\, F^{l}{}_{k}\,A_{i}\right)\right] \qquad \text{(D.8)}
\end{aligned}
$$

Adding all terms together one arrives at

$$
\begin{aligned}
R_{ij} \quad = \quad & \tilde{R}_{ij} - \frac{1}{2}\,\nabla_{k}\left(F^{k}{}_{i}\,A_{j}\right) \,-\, \frac{1}{2}\,\nabla_{k}\left(F^{k}{}_{j}\,A_{i}\right) \\
& + \frac{1}{4}\,F^{k}{}_{i}\,B_{kj} \,-\, \frac{1}{2}\,\tilde{\Gamma}^{l}{}_{(kj)}\,A_{l}\,F^{k}{}_{i} \\
& + \frac{1}{4}\,F^{k}{}_{j}\,B_{ki} \,-\, \frac{1}{2}\,\tilde{\Gamma}^{l}{}_{(ik)}\,A_{l}\,F^{k}{}_{j} \\
& + \frac{1}{4}\,F_{lk}\,F^{lk}\,A_{i}\,A_{j}\ . \qquad \text{(D.9)}
\end{aligned}
$$

Using now

$$
\begin{aligned}
\nabla_{k}\left(F^{k}{}_{i}\,A_{j}\right) \quad &= \quad \left(\nabla_{k}\,F^{k}{}_{i}\right)\,A_{j} \,+\, F^{k}{}_{i}\nabla_{k}\,A_{j} \;=\; \left(\nabla_{k}\,F^{k}{}_{i}\right)\,A_{j} \,+\, F^{k}{}_{i}\,\left(\partial_{k}\,A_{j} \,-\, \tilde{\Gamma}^{l}{}_{(kj)}\,A_{l}\right) \\
&= \quad \left(\nabla_{k}\,F^{k}{}_{i}\right)\,A_{j} \,+\, F^{k}{}_{i}\,\left(\partial_{[k}\,A_{j]} \,+\, \partial_{(k}\,A_{j)}\right) \,-\, \tilde{\Gamma}^{l}{}_{(kj)}\,A_{l}\,F^{k}{}_{i} \\
&= \quad \left(\nabla_{k}\,F^{k}{}_{i}\right)\,A_{j} \,+\, \frac{1}{2}\,F^{k}{}_{i}\,\left(\,F_{kj} \,+\, B_{kj}\,\right) \,-\, \tilde{\Gamma}^{l}{}_{(kj)}\,A_{l}\,F^{k}{}_{i} \qquad \text{(D.10)}
\end{aligned}
$$

to rewrite eq. (D.9) one obtains exactly the expression for R_{ij} given in (D.1).

Performing the last step and computing the Ricci scalar one obtains

$$
\begin{aligned}
R_{(11)} \quad &= \quad g^{MN}\,R_{MN} \;=\; g^{11\,11}\,R_{11\,11} \,+\, 2\,g^{11\mu}\,R_{11\mu} \,+\, g^{\mu\nu}\,R_{\mu\nu} \\
&= \quad \frac{1}{4}\,(1 + A_{\tau}A^{\tau})\,F_{\lambda\rho}F^{\lambda\rho} \;-\; 2\,A^{\mu}\,\left(\frac{1}{2}\,\nabla_{\nu}\,F_{\mu}{}^{\nu} \,+\, \frac{1}{4}\,A_{\mu}F_{\lambda\rho}F^{\lambda\rho}\right) \\
&+ \quad g^{\mu\nu}\,\left(\;\tilde{R}_{\mu\nu} \,+\, \frac{1}{2}\,F_{\mu}{}^{\rho}F_{\rho\nu} \,+\, \frac{1}{4}\,A_{\mu}\,A_{\nu}\,F_{\lambda\rho}F^{\lambda\rho} \,+\, \frac{1}{2}\,\left(\,A_{\nu}\,\nabla_{\rho}\,F_{\mu}{}^{\rho} \,+\, A_{\mu}\,\nabla_{\rho}\,F_{\nu}{}^{\rho}\,\right)\;\right)
\end{aligned}
$$

$$
R_{(11)} \quad = \quad R_{(10)} \;-\; \frac{1}{4}\,F_{\mu\nu}F^{\mu\nu}\ . \qquad \text{(D.11)}
$$

The 3. Term

$$\begin{aligned} \mathit{3.Term} &= 2\cdot\begin{pmatrix}4\\2\end{pmatrix}\cdot g^{11\,b_1}\,g^{a_2\,11}\,g^{a_3b_3}\,g^{a_4b_4}\,G_{11\,a_2a_3a_4}G_{b_1\,11\,b_3b_4} \\ &= 2\cdot\begin{pmatrix}4\\2\end{pmatrix}\cdot(-A^{b_1})\,(-A^{a_2})\,g^{a_3b_3}\,g^{a_4b_4}\,(-H_{a_2a_3a_4})\,H_{b_1b_3b_4} \\ &= -\,12\,A^{b_1}\,A^{a_2}\,H_{a_2}{}^{b_3b_4}\,H_{b_1b_2b_3} \end{aligned}$$

There are 6 possibilities to choose two metric factors out of the four $g^{\alpha_i\beta_i}$. And then an additional factor of 2 due to the two distributions of the two 11's on them.

A tensor identity

$$\begin{aligned} A_{[a_1}H_{a_2a_3a_4]}\,A^{[a_1}H^{a_2a_3a_4]} &= \frac{1}{4}\,(\,A_{a_1}H_{a_2a_3a_4}\,-\,A_{a_2}H_{a_1a_3a_4}\,-\,A_{a_3}H_{a_2a_1a_4}\,-\,A_{a_4}H_{a_2a_3a_1}) \\ &\quad\ \frac{1}{4}\,(\,A^{a_1}H^{a_2a_3a_4}\,-\,A^{a_2}H^{a_1a_3a_4}\,-\,A^{a_3}H^{a_2a_1a_4}\,-\,A^{a_4}H^{a_2a_3a_1}) \\ &= \frac{1}{4^2}\Big[\ (A^{(1)})^2(H^{(3)})^2\ -\ 2\,A_{a_1}\,A^{a_2}\,H_{a_2a_3a_4}\,H^{a_1a_3a_4} \\ &\qquad\ (A^{(1)})^2(H^{(3)})^2\ -\ 2\,A_{a_1}\,A^{a_3}\,H_{a_2a_3a_4}\,H^{a_2a_1a_4} \\ &\qquad\ (A^{(1)})^2(H^{(3)})^2\ -\ 2\,A_{a_1}\,A^{a_4}\,H_{a_2a_3a_4}\,H^{a_2a_3a_1} \\ &\qquad\ (A^{(1)})^2(H^{(3)})^2\ +\ 2\,A_{a_2}\,A^{a_3}\,H_{a_1a_3a_4}\,H^{a_2a_1a_4} \\ &\qquad\qquad\qquad\qquad\ +\ 2\,A_{a_2}\,A^{a_4}\,H_{a_1a_3a_4}\,H^{a_2a_3a_1} \\ &\qquad\qquad\qquad\qquad\ +\ 2\,A_{a_3}\,A^{a_4}\,H_{a_2a_1a_4}\,H^{a_2a_3a_1}\ \Big] \\ &= \frac{1}{4}\,(A^{(1)})^2(H^{(3)})^2\ -\ \frac{12}{4^2}\,A_a\,A^b\,H_{bcd}\,H^{acd} \end{aligned}$$

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